Introduction to Network Analysis using Pajek

3. Structure of Networks 1

Subnetworks

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Outline

1. Size of networks
2. Pajek
3. Statistics
4. Homomorphisms
5. Partitions
6. Subgraphs
7. Cuts

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Current version of slides (April 17, 2018 at 02:41):
Degrees

degree of node \( v \), \( \text{deg}(v) = \) number of links with \( v \) as an endnode;
indegree of node \( v \), \( \text{indeg}(v) = \) number of links with \( v \) as a terminal node (endnode is both initial and terminal);
outdegree of node \( v \), \( \text{outdeg}(v) = \) number of links with \( v \) as an initial node.

initial node \( v \) ⇔ \( \text{indeg}(v) = 0 \)
terminal node \( v \) ⇔ \( \text{outdeg}(v) = 0 \)

\[ n = 12, \ m = 23, \ \text{indeg}(e) = 3, \ \text{outdeg}(e) = 5, \ \text{deg}(e) = 6 \]

\[ \sum_{v \in V} \text{indeg}(v) = \sum_{v \in V} \text{outdeg}(v) = |A| + 2|E| - |E_0|, \ \sum_{v \in V} \text{deg}(v) = 2|L| - |L_0| \]
The size of a network/graph is expressed by two numbers: number of nodes $n = |V|$ and number of links $m = |L|$.

In a simple undirected graph (no parallel edges, no loops) $m \leq \frac{1}{2}n(n - 1)$; and in a simple directed graph (no parallel arcs) $m \leq n^2$.

Small networks (some tens of nodes) – can be represented by a picture and analyzed by many algorithms (UCINET, NetMiner). Also middle size networks (some hundreds of nodes) can still be represented by a picture (!?), but some analytical procedures can’t be used.

Till 1990 most networks were small – they were collected by researchers using surveys, observations, archival records, . . . The advances in IT allowed to create networks from the data already available in the computer(s). Large networks became reality. Large networks are too big to be displayed in details; special algorithms are needed for their analysis (Pajek).
Large networks

Large network – several thousands or millions of nodes. Can be stored in computer’s memory – otherwise huge network. 64-bit computers!

Jure Leskovec: SNAP – Stanford Large Network Dataset Collection

Social networks

<table>
<thead>
<tr>
<th>Name</th>
<th>Type</th>
<th>Nodes</th>
<th>Edges</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ego-Facebook</td>
<td>Undirected</td>
<td>4,039</td>
<td>88,234</td>
<td>Social circles from Facebook (anonymized)</td>
</tr>
<tr>
<td>ego-Gplus</td>
<td>Directed</td>
<td>107,614</td>
<td>13,673,453</td>
<td>Social circles from Google+</td>
</tr>
<tr>
<td>ego-Twitter</td>
<td>Directed</td>
<td>381,306</td>
<td>1,758,140</td>
<td>Social circles from Twitter</td>
</tr>
<tr>
<td>soc-Epinions1</td>
<td>Directed</td>
<td>75,879</td>
<td>508,837</td>
<td>Who-trusts-whom network of Epinions.com</td>
</tr>
<tr>
<td>soc-LiveJournal1</td>
<td>Directed</td>
<td>4,847,571</td>
<td>63,393,773</td>
<td>LiveJournal online social network</td>
</tr>
<tr>
<td>soc-Pokec</td>
<td>Directed</td>
<td>1,532,803</td>
<td>30,622,554</td>
<td>Pokec online social network</td>
</tr>
<tr>
<td>soc-Slashdot0811</td>
<td>Directed</td>
<td>77,360</td>
<td>905,468</td>
<td>Slashdot social network from November 2008</td>
</tr>
<tr>
<td>soc-Slashdot0922</td>
<td>Directed</td>
<td>82,168</td>
<td>948,464</td>
<td>Slashdot social network from February 2009</td>
</tr>
<tr>
<td>wiki-Vote</td>
<td>Directed</td>
<td>7,115</td>
<td>103,689</td>
<td>Wikipedia who-votes-on-whom network</td>
</tr>
</tbody>
</table>

Networks with ground-truth communities

<table>
<thead>
<tr>
<th>Name</th>
<th>Type</th>
<th>Nodes</th>
<th>Edges</th>
<th>Communities</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>com-LiveJournal</td>
<td>Undirected, Communities</td>
<td>3,997,962</td>
<td>34,681,189</td>
<td>287,512</td>
<td>LiveJournal online social network</td>
</tr>
<tr>
<td>com-Friendster</td>
<td>Undirected, Communities</td>
<td>65,508,366</td>
<td>1,806,087,135</td>
<td>957,154</td>
<td>Friendster online social network</td>
</tr>
<tr>
<td>com-Orkut</td>
<td>Undirected, Communities</td>
<td>3,072,441</td>
<td>117,185,083</td>
<td>6,286,363</td>
<td>Orkut online social network</td>
</tr>
</tbody>
</table>

Pajek datasets.
Average degree $\bar{d} = \frac{1}{n} \sum_{v \in V} \deg(v) = \frac{2m}{n}$. Most real-life large networks are sparse – the number of nodes and links are of the same order. This property is also known as a Dunbar’s number.

The basic idea is that if each node has to spend for each link certain amount of "energy" to maintain the links to selected other nodes then, since it has a limited "energy" at its disposal, the number of links should be limited. In human networks the Dunbar’s number is between 100 and 150.
Let us look to time complexities of some typical algorithms:

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Time Complexity</th>
<th>1.000</th>
<th>10.000</th>
<th>100.000</th>
<th>1.000.000</th>
<th>10.000.000</th>
</tr>
</thead>
<tbody>
<tr>
<td>LinAlg</td>
<td>$O(n)$</td>
<td>0.00 s</td>
<td>0.015 s</td>
<td>0.17 s</td>
<td>2.22 s</td>
<td>22.2 s</td>
</tr>
<tr>
<td>LogAlg</td>
<td>$O(n \log n)$</td>
<td>0.00 s</td>
<td>0.06 s</td>
<td>0.98 s</td>
<td>14.4 s</td>
<td>2.8 m</td>
</tr>
<tr>
<td>SqrtAlg</td>
<td>$O(\sqrt{n})$</td>
<td>0.01 s</td>
<td>0.32 s</td>
<td>10.0 s</td>
<td>5.27 m</td>
<td>2.78 h</td>
</tr>
<tr>
<td>SqrAlg</td>
<td>$O(n^2)$</td>
<td>0.07 s</td>
<td>7.50 s</td>
<td>12.5 m</td>
<td>20.8 h</td>
<td>86.8 d</td>
</tr>
<tr>
<td>CubAlg</td>
<td>$O(n^3)$</td>
<td>0.10 s</td>
<td>1.67 m</td>
<td>1.16 d</td>
<td>3.17 y</td>
<td>3.17 ky</td>
</tr>
</tbody>
</table>

For the interactive use on large graphs already quadratic algorithms, $O(n^2)$, are too slow.
In analysis of a large network (several thousands or millions of nodes, the network can be stored in computer memory) we can’t display it in its totality; also there are only few algorithms available.

To analyze a large network we can use statistical approach or we can identify smaller (sub) networks that can be analyzed further using more sophisticated methods.
In Pajek analysis and visualization are performed using 6 data types:

- **network** (graph),
- **partition** (nominal or ordinal properties of nodes),
- **vector** (numerical properties of nodes),
- **cluster** (subset of nodes),
- **permutation** (reordering of nodes, ordinal properties), and
- **hierarchy** (general tree structure on nodes).

Pajek supports also **multi-relational**, **temporal** and **two-mode** networks.
The power of Pajek is based on several transformations that support different transitions among these data structures. Also the menu structure of the main Pajek’s window is based on them. Pajek’s main window uses a ‘calculator’ paradigm with list-accumulator for each data type. The operations are performed on the currently active (selected) data and are also returning the results through accumulators.

The procedures are available through the main window menus. Frequently used sequences of operations can be defined as macros. This allows also the adaptations of Pajek to groups of users from different areas (social networks, chemistry, genealogy, computer science, mathematics…) for specific tasks. Pajek supports also repetitive operations on series of networks.
Input data

- numeric $\rightarrow$ vector
- ordinal $\rightarrow$ permutation
- nominal $\rightarrow$ clustering (partition)

Computed properties

global: number of nodes, edges/arcs, components; maximum core number, ...  
local: degrees, cores, indices (betweenness, hubs, authorities, ...)  
inspections: partition, vector, values of lines, ...

Associations between computed (structural) data and input (measured) data.
The global computed properties are reported by Pajek’s commands or can be seen using the Info option. In repetitive commands they are stored in vectors.

The local properties are computed by Pajek’s commands and stored in vectors or partitions. To get information about their distribution use the Info option.

As an example, let us look at The Edinburgh Associative Thesaurus network. The EAT is a network of word association as collected from subjects (students). The weight on the arcs is the count of word associations.

File/Network/Read eatRS.net
Info/Network/General

It has 23219 nodes and 325624 arcs (564 loops); number of links with value=1 is 227481.
To identify the nodes with the largest degree:

Net/Partitions/Degree/All
Partition/Make vector
Info/Vector +10

The largest degrees have the nodes:

<table>
<thead>
<tr>
<th>vertex</th>
<th>deg</th>
<th>label</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12720</td>
<td>1108 ME</td>
</tr>
<tr>
<td>2</td>
<td>12459</td>
<td>1074 MAN</td>
</tr>
<tr>
<td>3</td>
<td>8878</td>
<td>878 GOOD</td>
</tr>
<tr>
<td>4</td>
<td>18122</td>
<td>875 SEX</td>
</tr>
<tr>
<td>5</td>
<td>13793</td>
<td>803 NO</td>
</tr>
<tr>
<td>6</td>
<td>13181</td>
<td>799 MONEY</td>
</tr>
<tr>
<td>7</td>
<td>23136</td>
<td>732 YES</td>
</tr>
<tr>
<td>8</td>
<td>15080</td>
<td>723 PEOPLE</td>
</tr>
<tr>
<td>9</td>
<td>13948</td>
<td>720 NOTHING</td>
</tr>
<tr>
<td>10</td>
<td>22973</td>
<td>716 WORK</td>
</tr>
</tbody>
</table>
Pajek (0.89 and higher) supports interaction with statistical program R and the use of other external programs as tools (menu Tools). In Pajek we determine the degrees of nodes and submit them to R:

```
Network/Info/General
Network/Create Vector/Centrality/Degree/All
Tools/R/Send to R/Current Vector
```

In R we determine their distribution and plot it:

```
summary(v2)
t <- table(v2)
x <- as.numeric(names(t))
plot(x, t, log = 'xy', main = 'degree distribution',
    xlab = 'deg', ylab = 'freq')
```

The obtained picture can be saved with File/Save as in selected format (PDF or PS for \LaTeX; Windows metafile format for inclusion in Word).

Attention! The nodes of degree 0 make problems with log = 'xy'.
EAT all-degree distribution
Erdős and Renyi defined a **random graph** as follows: every possible link is included in a graph with a given probability $p$. In Pajek

Network/Create
Random Network/
Bernoulli/Poisson/Undirected/
General [100] [2.5]

instead of probability $p$ a more intuitive average degree is used

$$\overline{\text{deg}} = \frac{1}{n} \sum_{v \in V} \text{deg}(v)$$

It holds $p = \frac{m}{m_{\text{max}}}$ and, for simple graphs, also $\overline{\text{deg}} = \frac{2m}{n}$.

Random graph in the picture has 100 nodes and average degree 3.
Real-life networks are usually not random in the Erdős/Renyi sense. The analysis of their distributions gave a new view about their structure – Watts (Small worlds), Barabási (nd/networks, Linked).
We read in Pajek the citation network cite.net. First we remove loops and multiple links. Then we determine the indegrees and outdegrees and call R from Pajek submitting all vectors.

```r
R called from Pajek
The following vectors read:
 v3 : From partition 1 (548600)
 v4 : From partition 2 (548600)

> inTab <- table(v3)
> indeg <- as.integer(names(inTab))
> inDeg <- indeg[indeg>0]
> inFreq <- as.vector(inTab[indeg>0])
> plot(inDeg,inFreq,log='xy',main="in-degree distribution")
> ouTab <- table(v4)
> outdeg <- as.integer(names(ouTab))
> outDeg <- outdeg[outdeg>0]
> outFreq <- as.vector(ouTab[outdeg>0])
> plot(outDeg,outFreq,log='xy',main="out-degree distribution")
```
The in-degree distribution is "scale-free"-like. The parameters can be determined using the package of Clauset, Shalizi and Newman. See also Stumpf, et al.: Critical Truths About Power Laws.
From the file `Year.clu` containing the year of publication of a paper we can get the distribution of papers by years. For the centrality network we get:

```r
> setwd("C:/Users/Batagelj/work/Python/WoS/Central")
> years <- read.table(file="Year.clu",header=FALSE,skip=2)$V1
> t <- table(years)
> year <- as.integer(names(t))
> freq <- as.vector(t[1950<=year & year<=2009])
> y <- 1950:2009
> plot(y,freq)
> model <- nls(freq~c*dlnorm(2010-y,a,b),start=list(c=350000,a=2,b=0.7))
> model
Call:  nls(formula = freq ~ c * dlnorm(2010 - y, a, b),
       start = list(c = 350000, a = 2, b = 0.7))
Coefficients:
          Estimate Std. Error t value
(Intercept)   350000     24910  139.13
           a    2.4910     0.249    9.96
           b   0.6624     0.066   10.02
Residual standard error: 20474181 on 58 degrees of freedom
Number of iterations to convergence: 7
Achieved convergence tolerance: 3.978e-06
> lines(y,predict(model,list(x=2010-y)),col='red')
```

It can be well approximated by the lognormal distribution, but also by the generalized reciprocal power exponential curve $c * (x + d)^{\frac{a}{b+x}}$. 
Papers by years / centrality network
Homomorphisms of graphs

Functions \((\varphi, \psi)\), \(\varphi: V \rightarrow V'\) and \(\psi: L \rightarrow L'\) determine a weak homomorphism of graph \(G = (V, L)\) in graph \(H = (V', L')\) iff:

\[
\forall u, v \in V \forall p \in L : (p(u : v) \Rightarrow \psi(p)(\varphi(u) : \varphi(v)))
\]

and they determine a (strong) homomorphism of graph \(G\) in graph \(H\) iff:

\[
\forall u, v \in V \forall p \in L : (p(u, v) \Rightarrow \psi(p)(\varphi(u), \varphi(v)))
\]

If \(\varphi\) and \(\psi\) are bijections and the condition hold in both direction we get an isomorphism of graphs \(G\) and \(H\). We denote the weak isomorphism by \(G \sim H\); and the (strong) isomorphism by \(G \approx H\). It holds \(\approx \subset \sim\).

An invariant of graph is called each graph characteristic that has the same value for all isomorphic graphs.
Homomorphism

\[\varphi\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9\\t&y&z&x&v&u&z&y&t\end{array}\]
\[\psi\begin{array}{cccccccccccccc}a&b&c&d&e&f&g&h&i&j&k&l&m\\E&J&D&H&G&C&H&G&B&F&J&I&E\end{array}\]

homoEna.net
Isomorphic graphs

\[ \varphi \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ b & h & j & a & g & c & e & i & d & f \end{array} \]

izoPet.net
A nonempty subset \( C \subseteq V \) is called a cluster (group). A nonempty set of clusters \( C = \{C_i\} \) forms a clustering. Clustering \( C = \{C_i\} \) is a partition iff

\[
\bigcup_i C_i = V \quad \text{and} \quad i \neq j \Rightarrow C_i \cap C_j = \emptyset
\]

Clustering \( C = \{C_i\} \) is a hierarchy iff

\[
C_i \cap C_j \in \{\emptyset, C_i, C_j\}
\]

Hierarchy \( C = \{C_i\} \) is complete, iff \( \bigcup C = V \); and is basic if for all \( v \in \bigcup C \) also \( \{v\} \in C \).
Examples

Node set:
\[ \mathcal{V} = \{a, b, c, d, e, f, g\} \]

Partition:
\[ \mathbf{C} = \{\{a, b, e\}, \{c, g\}, \{d, f\}\} \]

Cluster, class:
\[ C_2 = \{c, g\} \]

Hierarchy:
\[ \mathbf{H} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{a, e\}, \{c, g\}, \{d, f\}, \{a, b, e\}, \{c, d, f, g\}, \{a, b, c, d, e, f, g\}\} \]
Draw / Partition

Draw/Network + First Partition
Layout/Energy/Kamada-Kawai/Free
Layout/Energy/Fruchterman Reingold/2D
Contraction of cluster

Contraction of cluster \( C \) is called a graph \( \mathcal{G}/C \), in which all nodes of the cluster \( C \) are replaced by a single node, say \( c \). More precisely:

\[
\mathcal{G}/C = (\mathcal{V}', \mathcal{L}') \text{, where } \mathcal{V}' = (\mathcal{V} \setminus C) \cup \{c\} \text{ and } \mathcal{L}' \text{ consists of links from } \mathcal{L} \text{ that have both endnodes in } \mathcal{V} \setminus C. \text{ Beside these it contains also a 'star' with the center } c \text{ and: arc } (v, c), \text{ if } \exists p \in \mathcal{L}, u \in C : p(v, u); \text{ or arc } (c, v), \text{ if } \exists p \in \mathcal{L}, u \in C : p(u, v). \]

There is a loop \((c, c)\) in \( c \) if \( \exists p \in \mathcal{L}, u, v \in C : p(u, v) \).

In a network over graph \( \mathcal{G} \) we have also to specify how are determined the values/weights in the shrunk part of the network. Usually as the sum or maximum/minimum of the original values.
Snyder and Kick’s international trade. Matrix display of dense networks.

\[ w(C_i, C_j) = \frac{n(C_i, C_j)}{n(C_i) \cdot n(C_j)} \]

Macros.
Computing the weights $w$

File/Pajek Project File/Read [SKtrade.paj]
Network/Create New Network/Transform/Remove/Loops [No]
Network/Create New Network/Transform/Edges -> Arcs [No]
Operations/Network+Partition/Shrink Network [1 0]

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>Label</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2</td>
<td>30</td>
<td>13</td>
<td>56</td>
<td>42</td>
<td>45</td>
<td>4</td>
<td>#usa</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>74</td>
<td>25</td>
<td>196</td>
<td>20</td>
<td>37</td>
<td>12</td>
<td>#cub</td>
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<tr>
<td>3</td>
<td>12</td>
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<td>124</td>
<td>16</td>
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<td>5</td>
<td>#per</td>
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<td>55</td>
<td>217</td>
<td>130</td>
<td>694</td>
<td>427</td>
<td>483</td>
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<tr>
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<td>42</td>
<td>8</td>
<td>14</td>
<td>406</td>
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<td>117</td>
<td>11</td>
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<tr>
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<td>37</td>
<td>43</td>
<td>444</td>
<td>142</td>
<td>307</td>
<td>30</td>
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</tr>
<tr>
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<td>4</td>
<td>5</td>
<td>39</td>
<td>9</td>
<td>30</td>
<td>2</td>
<td>#aut</td>
</tr>
</tbody>
</table>

Partition/Make Permutation
[select partition (Sub)continents]
Operations/Partition+Permutation/
   Functional Composition Partition*Permutation
Partition/Count

count    2  15  7  29  33  30  2
... Computing the weights $w$

Partition/Copy to Vector
Vector/Create Constant Vector [7 1.0]
[select as second vector Copy of partition ...]
Vectors/Divide (First/Second)
Network/Create Vector/Get Loops
Vectors/Add (First+Second)
Operations/Network+Vector/Transform/Put Loops/as Arcs
[select vector Divide V? by ...]
Operations/Network+Vector/Vector#Network/input
Operations/Network+Vector/Vector#Network/output

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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</thead>
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<td>1.00</td>
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<td>0.97</td>
<td>0.64</td>
<td>0.75</td>
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<td></td>
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<tr>
<td>#cub</td>
<td>1.00</td>
<td>0.33</td>
<td>0.24</td>
<td>0.45</td>
<td>0.04</td>
<td>0.08</td>
<td>0.40</td>
</tr>
<tr>
<td>#per</td>
<td>0.86</td>
<td>0.27</td>
<td>0.67</td>
<td>0.61</td>
<td>0.07</td>
<td>0.17</td>
<td>0.36</td>
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<tr>
<td>#uki</td>
<td>0.95</td>
<td>0.50</td>
<td>0.64</td>
<td>0.83</td>
<td>0.45</td>
<td>0.56</td>
<td>0.71</td>
</tr>
<tr>
<td>#mli</td>
<td>0.64</td>
<td>0.02</td>
<td>0.06</td>
<td>0.42</td>
<td>0.11</td>
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<tr>
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<td>0.20</td>
<td>0.51</td>
<td>0.14</td>
<td>0.34</td>
<td>0.50</td>
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<td>0.36</td>
<td>0.67</td>
<td>0.14</td>
<td>0.50</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Note: Set diagonal values to 1?

Macro weights.
A **subgraph** \( \mathcal{H} = (\mathcal{V}', \mathcal{L}') \) of a given graph \( \mathcal{G} = (\mathcal{V}, \mathcal{L}) \) is a graph which set of links is a subset of set of links of \( \mathcal{G} \), \( \mathcal{L}' \subseteq \mathcal{L} \), its node set is a subset of set of nodes of \( \mathcal{G} \), \( \mathcal{V}' \subseteq \mathcal{V} \), and it contains all endnodes of \( \mathcal{L}' \).

A subgraph can be **induced** by a given subset of nodes or links. It is a **spanning** subgraph iff \( \mathcal{V}' = \mathcal{V} \).

To obtain a **subnetwork** also the properties/weights have to be restricted to \( \mathcal{V}' \) and \( \mathcal{L}' \).
Cut-out – induced subgraph: Snyder and Kick
Africa

Cut-out: Snyder and Kick
Latin America: South America

Operations/Network + Partition/Extract Subnetwork [3,4]
Operations/Network + Partition/Transform/Remove lines/Inside clusters [3,4]

The nodes can be manually put on a rectangular grid produced by

[Draw] Move/Grid
The standard approach to find interesting groups inside a network is based on properties/weights – they can be measured or computed from network structure.

The node-cut of a network $\mathcal{N} = (\mathcal{V}, \mathcal{L}, p)$, $p : \mathcal{V} \rightarrow \mathbb{R}$, at selected level $t$ is a subnetwork $\mathcal{N}(t) = (\mathcal{V}', \mathcal{L}(\mathcal{V}'), p)$, determined by the set

$$\mathcal{V}' = \{ v \in \mathcal{V} : p(v) \geq t \}$$

and $\mathcal{L}(\mathcal{V}')$ is the set of links from $\mathcal{L}$ that have both endnodes in $\mathcal{V}'$.

The link-cut of a network $\mathcal{N} = (\mathcal{V}, \mathcal{L}, w)$, $w : \mathcal{L} \rightarrow \mathbb{R}$, at selected level $t$ is a subnetwork $\mathcal{N}(t) = (\mathcal{V}(\mathcal{L}'), \mathcal{L}', w)$, determined by the set

$$\mathcal{L}' = \{ e \in \mathcal{L} : w(e) \geq t \}$$

and $\mathcal{V}(\mathcal{L}')$ is the set of all endnodes of the links from $\mathcal{L}'$. 
Each node represents a company that competes in the Internet industry, 1998 to 2001. $n = 219$, $m = 631$. Red – content, blue – infrastructure, green – commerce. Two companies are linked with an edge if they have announced a joint venture, strategic alliance or other partnership.
Let $\mathcal{G}$ be a simple undirected graph. A **triangular network** $\mathcal{N}_T(\mathcal{G}) = (\mathcal{V}, \mathcal{E}_T, w)$ determined by $\mathcal{G}$ is a subgraph $\mathcal{G}_T = (\mathcal{V}, \mathcal{E}_T)$ of $\mathcal{G}$ which set of edges $\mathcal{E}_T$ consists of all triangular edges of $\mathcal{E}(\mathcal{G})$. For $e \in \mathcal{E}_T$ the weight $w(e)$ equals to the number of different triangles in $\mathcal{G}$ to which $e$ belongs.

Triangular networks can be used to efficiently identify dense clique-like parts of a graph. If an edge $e$ belongs to a $k$-clique in $\mathcal{G}$ then $w(e) \geq k - 2$. 

Network/Create New Network/with Ring Counts/3-Rings
Link-cut: Krebs Internet Industries, $w_3 \geq 5$
Overlap weight – definition

The (topological) **overlap weight** of an edge \( e = (u : v) \in \mathcal{E} \) in an undirected simple graph \( G = (\mathcal{V}, \mathcal{E}) \) is defined as

\[
o(e) = \frac{t(e)}{(\deg(u) - 1) + (\deg(v) - 1) - t(e)}
\]

\( t(e) = w_3(e) \) is the number of triangles (cycles of length 3) to which the edge \( e \) belongs. In the case \( \deg(u) = \deg(v) = 1 \) we set \( o(e) = 0 \).

The overlap weight is essentially a Jaccard similarity index

\[
J(X, Y) = \frac{|X \cap Y|}{|X \cup Y|}
\]

for \( X = N(u) \setminus \{v\} \) and \( Y = N(v) \setminus \{u\} \) where \( N(z) \) is the set of neighbors of a node \( z \).

Denoting \( \mu = \max_{e \in \mathcal{E}} t(e) \) and \( M(e) = \max(\deg(u), \deg(v)) - 1 \) we define a corrected overlap weight as

\[
o'(e) = \frac{t(e)}{\mu + M(e) - t(e)}
\]
Cuts in Pajek

The threshold value $t$ is determined on the basis of distribution of values of weight $w$ or property $p$. Usually we are interested in cuts that are not too large, but also not trivial.

Node-cut: $p$ stored in a vector

```
Vector/Info [+10] [#10]
Vector/Make Partition/by Intervals/Selected Thresholds [t]
Operations/Network + Partition/Extract Subnetwork [2]
```

Link-cut: weighted network

```
Network/Info/Line values [#10]
Network/Create New Network/Transform/Remove/Lines with Value/ lower than [t]
Network/Create Partition/Degree/All
Operations/Network + Partition/Extract Subnetwork [1-*]
```
Simple analysis using cuts

We look at the components of $N(t)$. Their number and sizes depend on $t$. Usually there are many small components. Often we consider only components of size at least $k$ and not exceeding $K$. The components of size smaller than $k$ are discarded as 'noninteresting'; and the components of size larger than $K$ are cut again at some higher level.

The values of thresholds $t$, $k$ and $K$ are determined by inspecting the distribution of node/link-values and the distribution of component sizes and considering additional knowledge on the nature of network or goals of analysis.

We developed some new and efficiently computable properties/weights.
The citation network analysis started in 1964 with the paper of Garfield et al. In 1989 Hummon and Doreian proposed three indices – weights of arcs that are proportional to the number of different source-sink paths passing through the arc. We developed algorithms to efficiently compute these indices.

Main subnetwork (arc-cut at level 0.007) of the SOM (self-organizing maps) citation network (4470 nodes, 12731 arcs). See paper.