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Semirings and Matrix Analysis of Networks

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Synonyms

Algebraic path problem; Matrix; Multiplication of vector and matrix; Network; Network multiplication; Semiring; Simple walk; Value matrix; Walk

Glossary

Algebraic structure:	A set with one or more operations defined on it and rules that hold for them
Network analysis:	A study of networks as representations of relations between discrete objects
Sparse matrix:	A matrix with most of entries equal to zero
Large network:	A network with several thousands or millions of nodes
Complete graph:	K_n – A graph in which every pair of nodes is linked

Definition

A network can be represented also with a corresponding matrix. Using matrix operations (addition and multiplication) over an appropriate semiring a unified approach to several network analysis problems can be developed. Matrix multiplication is about traveling on network.

Introduction

Semirings are algebraic structures with two operations that provide the basic conditions for studying matrix addition and multiplication and path problems in networks. Several results and algorithms from different fields of application turn out to be special cases over the corresponding semirings.

Semirings

Let $\mathbb{K}$ be a set and *a, b, c* elements from $\mathbb{K}$. A semiring (Abdali and Saunders 1985; Baras and Theodorakopoulos 2010; Batagelj 1994; Carré 1979) is an algebraic structure $(\mathbb{K}, \oplus, \odot, 0, 1)$ with two binary operations (addition $\oplus$ and multiplication $\odot$) where:

- $(\mathbb{K}, \oplus, 0)$ is an abelian monoid with the neutral element 0 (zero):

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$$a \oplus b = b \oplus a \quad \text{commutativity}$$
$$(a \oplus b) \oplus c = a \oplus (b \oplus c) \quad \text{associativity}$$
$$a \oplus 0 = a \quad \text{existence of zero}$$

• $(\mathbb{K}, \odot, 1)$ is a monoid with the neutral element 1 (unit):

$$(a \odot b) \odot c = a \odot (b \odot c) \quad \text{associativity}$$
$$a \odot 1 = 1 \odot a = a \quad \text{existence of a unit}$$

• Multiplication $\odot$ distributes over addition $\oplus$:

$$a \odot (b \oplus c) = a \odot b \oplus a \odot c$$
$$(b \oplus c) \odot a = b \odot a \oplus c \odot a$$

In formulas we assume precedence of multiplication over addition.

A semiring $(\mathbb{K}, \oplus, \odot, 0, 1)$ is *complete* if the addition is well defined for countable sets of elements and the commutativity, associativity, and distributivity hold in the case of countable sets. These properties are generalized in this case; for example, the distributivity takes the form

$$\left(\bigoplus_i a_i\right) \odot \left(\bigoplus_j b_j\right) = \bigoplus_i \left(\bigoplus_j (a_i \odot b_j)\right) = \bigoplus_{i,j} (a_i \odot b_j).$$

The addition is *idempotent* if $a \oplus a = a$ for all $a \in \mathbb{K}$. In this case the semiring over a finite set $\mathbb{K}$ is complete.

A semiring $(\mathbb{K}, \oplus, \odot, 0, 1)$ is *closed* if for the additional (unary) *closure* operation * it holds for all $a \in \mathbb{K}$:

$$a^* = 1 \oplus a \odot a^* = 1 \oplus a^* \odot a.$$

The *power* a^n, $n \in \mathbb{N}$ of an element $a \in \mathbb{K}$ is defined by $a^0 = 1$ and $a^{n+1} = a^n \odot a$ for $n \geq 0$.

Different closures over the same semiring can exist. A complete semiring is always closed for the closure

$$a^* = \bigoplus_{k \in \mathbb{N}} a^k.$$

In a closed semiring we can also define a *strict closure* $\overline{a}$ as

$$\overline{a} = a \odot a^*.$$

In a semiring $(\mathbb{K}, \oplus, \odot, 0, 1)$ the *absorption law* holds if for all $a, b, c \in \mathbb{K}$:

$$a \odot b \oplus a \odot c \odot b = a \odot b.$$

Because of the distributivity, it is sufficient for the absorption law to check the property $1 \oplus c = 1$ for all $c \in \mathbb{K}$.

Combinatorial Semiring $(\mathbb{N}, +, \cdot, 0, 1)$

This is the most commonly used semiring. Also some other sets are used: $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{Q}$. For $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, the semiring is closed for $a^* = \sum_{k \in \overline{\mathbb{N}}} a^k$ because it is a complete semiring. An example of a closure for $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is $a^* = 1/(1 - a)$ for $a \neq 1, \infty$ and $0^* = 1$, $1^* = \infty$, and $\infty^* = \infty$. This semiring is commutative because it holds $a \odot b = b \odot a$ for all a and b in the set. Combinatorial semiring is not an idempotent semiring.

Reachability Semiring $(\{0, 1\}, \vee, \wedge, 0, 1)$

The logical (boolean, reachability) semiring is suitable for solving the connectivity questions in networks. The multiplication is commutative and the absorption law holds. The reachability semiring is closed for $a^* = 1 \vee a \wedge a^* = 1$.

Shortest Paths Semiring $(\overline{\mathbb{R}}_0^+, \min, +, \infty, 0)$

The commutativity of multiplication holds in this semiring. The semiring is closed for $a^* = \min(0, a + a^*) = 0$ (0 is the smallest element in the set $\overline{\mathbb{R}}_0^+$). Since min(0, a) = 0, the absorption law also holds. For the set $\overline{\mathbb{N}}$, the semiring is called a tropical semiring. Another set is $\overline{\mathbb{R}}$ and in this case the semiring is isomorphic ($x \rightarrow -x$) to max-plus semiring ($\mathbb{R} \cup \{-\infty\}$, max , + , $-\infty$, 0).

Pathfinder Semiring ($\overline{\mathbb{R}}_0^+$, min, $\boxed{r}$, ∞, 0)

The Pathfinder semiring (Schvaneveldt et al. 1988) is a special case from the family of semirings obtained as follows. Let $B \subseteq \overline{\mathbb{R}}$ be such that $(B, +, \cdot, 0, 1)$ or $(B, \min, +, U, 0)$ is a semiring ($U = \max(B)$). Therefore $0 \in B$ and $1 \in B$. Let $A \subseteq \mathbb{R}$ be such that $g : A \rightarrow B$ is a bijection. Let us define operations $\oplus$, ∇, $\odot$ so that g is a homomorphism:

$$g(a \oplus b) = g(a) + g(b),$$
$$g(a \nabla b) = \min(g(a), \ g(b)),$$
$$g(a \odot b) = g(a) \cdot g(b).$$

This is equivalent to

$$a \oplus b = g^{-1}(g(a) + g(b)),$$
$$a \nabla b = g^{-1}(\min(g(a), \ g(b))),$$
$$a \odot b = g^{-1}(g(a) \cdot g(b)).$$

The function g^{-1} is also a homomorphism. If g is strictly increasing function, then

$$a \nabla b = g^{-1}(\min(g(a), \ g(b))) = \min(a, b).$$

Since the homomorphisms preserve the algebraic properties, also the structures

$$(A, \oplus, \odot, g^{-1}(0), g^{-1}(1))$$

and

$$(A, \nabla, \oplus, g^{-1}(U), g^{-1}(0))$$

are semirings.

For g(:r) $= x^r$, $g^{-1}(y) = \sqrt[r]{y}$, we get the *Pathfinder semiring* ($\overline{\mathbb{R}}_0^+$, min, $\boxed{r}$, ∞, 0). The multiplicative operation is the *Minkowski operation* $a \boxed{r} b = \sqrt[r]{a^r + b^r}$. This semiring is closed for $a^* = 0$ and the absorption law holds in it.

In Pathfinder algorithm the value r for the Minkowski operation is selected according to a dissimilarity measure. For a value $r = 1$, the semiring is the shortest path semiring, and for a value $r = \infty$, the semiring is the min-max semiring.

More about semirings and several other examples can be found in (Baras and Theodorakopoulos 2010; Batagelj and Praprotnik 2016; Burkard et al. 1984; Carré 1979; Glazek 2002; Golan 1999; Gondran and Minoux 2008; Kepner and Gilbert 2011).

Matrices

An $m \times n$ matrix $\mathbf{A}$ over a set $\mathbb{K}$ is a rectangular array of elements from the set $\mathbb{K}$ that consists of m rows and n columns. The entry in the i-*th* row and j-th column is denoted by a_{ij}. If $m = n$ the matrix $\mathbf{A}$ is called a *square* matrix. The matrix with all entry values equal to 0 is called the *zero* matrix and is denoted by $\mathbf{0}_{mn}$.

The *transpose* of a matrix $\mathbf{A}$ is a matrix $\mathbf{A}^T$ in which the rows of $\mathbf{A}$ are written as the columns of $\mathbf{A}^T : a_{ij}^T = a_{ji}$. A square matrix $\mathbf{A}$ is *symmetric* if $\mathbf{A} = \mathbf{A}^T$.

A *diagonal matrix* is a square matrix $\mathbf{A}$ such that only diagonal elements are nonzero: $a_{ij} = 0$, for $i \neq j$. If $a_{ii} = 1$, $i = 1, \ldots, n$, a diagonal matrix is called the *identity* matrix I_n of order n. A square matrix $\mathbf{A}$ is *upper triangular* if $a_{ij} = 0$, $i > j$, and its transpose is a *lower triangular* matrix.

Let $\mathcal{M}_{mn}(\mathbb{K})$ be a set of matrices of order $m \times n$ over the semiring ($\mathbb{K}$, $\oplus$, $\odot$, 0, 1) in which we additionally require

$$\forall a \in \mathbb{K} : a \odot 0 = 0 \odot a = 0,$$

and let $\mathcal{M}(\mathbb{K})$ be a set of all matrices over the $\mathbb{K}$. The operations $\oplus$ and $\odot$ can be extended to the $\mathcal{M}(\mathbb{K})$:

$$\mathbf{A}, \mathbf{B} \in \mathcal{M}_{mn}(\mathbb{K}) : \mathbf{A} \oplus \mathbf{B} = [a_{uv} \oplus b_{uv}] \in \mathcal{M}_{mn}(\mathbb{K})$$
$$\mathbf{A} \in \mathcal{M}_{mk}(\mathbb{K}),$$
$$\mathbf{B} \in \mathcal{M}_{kn}(\mathbb{K}) : \mathbf{A} \odot \mathbf{B} = \left[\oplus_{t=1}^{k} a_{ut} \odot b_{tv}\right] \in \mathcal{M}_{mn}(\mathbb{K}).$$

Then:

- $(\mathcal{M}_{mn}(\mathbb{K}), \oplus, \mathbf{0}_{mn})$ is an abelian monoid.
- $(\mathcal{M}_{n^2}(\mathbb{K}), \odot, \mathbf{I}_n)$ is a monoid.
- $(\mathcal{M}_{n^2}(\mathbb{K}), \oplus, \odot, 0_n, \mathbf{I}_n)$ is a semiring.

For matrices **A** and **B**, it holds

$$(\mathbf{A} \odot \mathbf{B})^T = \mathbf{B}^T \odot \mathbf{A}^T.$$

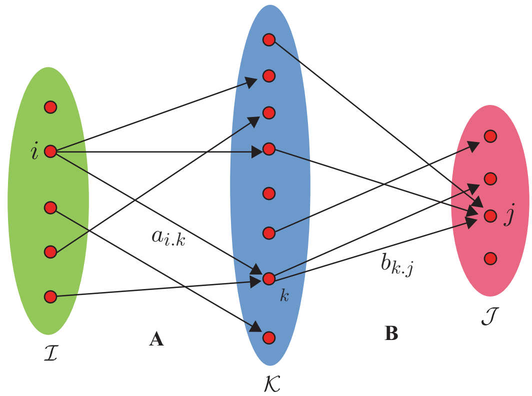
Network Multiplication

A (simple directed) network $\mathcal{N}$ is an ordered pair of sets $(\mathcal{V}, \mathcal{A})$ where $\mathcal{V}$ is the set of nodes and $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs (directed links). We assume that the set of nodes is finite $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{N} = ((\mathcal{I}, \mathcal{J}), \mathcal{A}, w)$ be a *simple two-mode network,* where $\mathcal{I}$ and $\mathcal{J}$ are disjoint (sub)sets of nodes ($\mathcal{V} = \mathcal{I} \cup \mathcal{J}$, $\mathcal{I} \cap \mathcal{J} = \emptyset$), $\mathcal{A}$ is a set of arcs linking $\mathcal{I}$ and $\mathcal{J}$, and the mapping w: $\mathcal{A} \to \mathbb{K}$ is the *arcs value function* also called a *weight.* We can assign to a network its *value matrix* $\mathbf{W} = [w_{ij}]$ with elements

$$w_{ij} = \begin{cases} w((i,j)) & (i,j) \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

The problem with value matrices in computer applications is their size. The value matrices of large networks are sparse. There is no need to store the zero values in a matrix, and different data structures can be used for saving and working with value matrices: special dictionaries and lists.

Let $\mathcal{N}_\mathbf{A} = ((\mathcal{I}, \mathcal{K}), \mathcal{A}_\mathbf{A}, w_\mathbf{A})$ and $\mathcal{N}_\mathbf{B} = ((\mathcal{K}, \mathcal{J}), \mathcal{A}_\mathbf{B}, w_\mathbf{B})$ be a pair of networks with corresponding matrices $\mathbf{A}_{\mathcal{I}\times\mathcal{K}}$ and $\mathbf{B}_{\mathcal{K}\times\mathcal{J}}$, respectively. Assume also that w_A: $\mathcal{A}_\mathbf{A} \to \mathbb{K}$, $w_\mathbf{B}$: $\mathcal{A}_\mathbf{B} \to \mathbb{K}$, and $(\mathbb{K}, \oplus, \odot, 0, 1)$ is a semiring. We say that such networks/matrices are *compatible.* The *product* $\mathcal{N}_\mathbf{A} * \mathcal{N}_\mathbf{B}$ of networks $\mathcal{N}_\mathbf{A}$ and $\mathcal{N}_\mathbf{B}$ is a network $\mathcal{N}_\mathbf{C} = ((\mathcal{I}, \mathcal{J}), \mathcal{A}_\mathbf{C}, w_\mathbf{C})$ for $\mathcal{A}_\mathbf{C} = \{(i,j); i \in \mathcal{I}, j \in \mathcal{J}, c_{ij} \neq 0\}$ and $w_\mathbf{C}((i,j)) = c_{ij}$ for $(i,j) \in \mathcal{A}_\mathbf{C}$, where $\mathbf{C} = [c_{ij}] = \mathbf{A} \odot \mathbf{B}$. If all three sets of nodes are the same ($\mathcal{I} = \mathcal{K} = \mathcal{J}$), we are dealing with ordinary one-mode networks (square matrices).



Semirings and Matrix Analysis of Networks, Fig. 1 Multiplication of networks

When do we get an arc in the product network? Let's look at the definition of the matrix product

$$c_{ij} = \bigoplus_{k \in \mathcal{K}} a_{ik} \odot b_{kj}.$$

There is an arc $(i, j) \in \mathcal{A}_\mathbf{C}$ if c_{ij} is nonzero. Therefore at least one term $a_{ik} \odot b_{kj}$ is nonzero, but this means that both a_{ik} and b_{kj} should be nonzero, and thus $(i,k) \in \mathcal{A}_\mathbf{A}$ and $(k,j) \in \mathcal{A}_\mathbf{B}$ (see Fig. 1):

$$c_{ij} = \bigoplus_{k \in N_\mathbf{A}(i) \cap N_\mathbf{B}^-(j)} a_{ik} \odot b_{kj},$$

where $N_\mathbf{A}(i)$ are the *successors* of node i in the network $\mathcal{N}_\mathbf{A}$ and $N_\mathbf{B}^-(j)$ are the *predecessors* of node j in the network $\mathcal{N}_\mathbf{B}$. The value of the entry c_{ij} equals to the value of all paths (of length 2) from $i \in \mathcal{I}$ to $j \in \mathcal{J}$ passing through some node $k \in \mathcal{K}$.

The standard procedure to compute the product of matrices $\mathbf{A}_{\mathcal{I}\times\mathcal{K}}$ and $\mathbf{B}_{\mathcal{K}\times\mathcal{J}}$ has the complexity $O(|\mathcal{I}| \cdot |\mathcal{K}| \cdot |\mathcal{J}|)$ and is therefore too slow to be used for large networks. Since the matrices of large networks are usually sparse, we can compute the product of two networks much faster considering only nonzero entries (Batagelj and Cerinšek 2013; Batagelj and Mrvar 2008):

for $k \in \mathrm{K}$ **do**
  for $i \in N^{-}_{\mathbf{A}}(k)$ **do**
    for $j \in N_{\mathbf{B}}(k)$ **do**
      If $\exists c_{ij}$ **then**
        $c_{ij} = c_{ij} \oplus a_{ik} \odot b_{kj}$
      else
        $c_{ij} = a_{ik} \odot b_{kj}$.

In general the multiplication of large sparse networks is a "dangerous" operation since the result can "explode" – it is not sparse.

From the network multiplication algorithm, we see that each intermediate node $k \in \mathcal{K}$ adds to a product network a complete two-mode subnetwork $K_{N^{-}_{\mathbf{A}}(k), N_{\mathbf{B}}(k)}$ (or, in the case $\mathbf{A} = \mathbf{B}$, a complete subnetwork $\mathrm{K}_{N(k)}$). If both degrees $\deg_{\mathbf{A}}(k) = |N^{-}_{\mathbf{A}}(k)|$ and $\deg_{\mathbf{B}}(k) = |N_{\mathbf{B}}(k)|$ are large, then already the computation of this complete subnetwork has a quadratic (time and space) complexity – the result "explodes."

If for the sparse networks $\mathcal{N}_{\mathbf{A}}$ and $\mathcal{N}_{\mathbf{B}}$, there are in $\mathcal{K}$ only few nodes with large degree and no one among them with large degree in both networks, then also the resulting product network $\mathcal{N}_{\mathbf{C}}$ is sparse.

The Algebraic Path Problem

The use of a special semiring and a multiplication of networks can lead us to the essence of the shortest path problem (Baras and Theodorakopoulos 2010). Many other network problems can be solved by replacing the usual addition and multiplication with the corresponding operations from an appropriate semiring.

Let $\mathcal{N} = (\mathcal{V}, \mathcal{A}, w)$ be a network where $w: \mathcal{A} \to \mathbb{K}$ is the value (weight) of arcs such that $(\mathbb{K}, \oplus, \odot, 0, 1)$ is a semiring. We denote the number of nodes as $n = |\mathcal{V}|$ and the number of arcs as $m = |\mathcal{A}|$.

A finite sequence of nodes $\sigma = (u_0, u_1, u_2, \ldots, u_{p-1}, u_p)$ is a *walk* of *length* p on $\mathcal{N}$ if every pair of neighboring nodes is linked: $(u_{i-1}, u_i) \in \mathcal{A}$, $i = 1, \ldots, p$. Finite sequence σ is a *semiwalk* or chain on $\mathcal{N}$ if every pair of neighboring nodes is linked neglecting the direction of an arc: $(u_{i-1}, u_j) \in \mathcal{A} \vee (u_i, u_{i-1}) \in \mathcal{A}, i = 1, \ldots, p$. A (semi)walk is closed if its end nodes coincide: $u_0 = u_p$. A walk is *simple* or a *path* if no node repeats in it. A closed walk with different nodes, except first and last, is called a *cycle*.

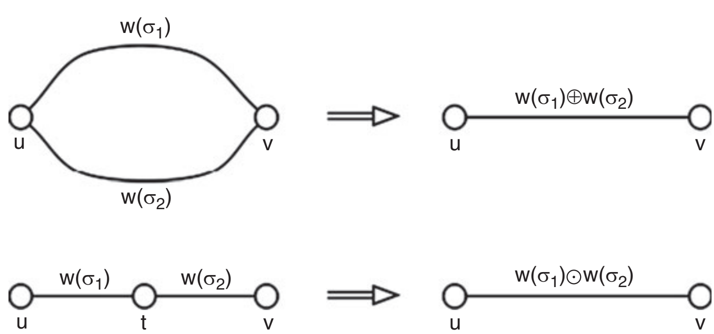
We can extend the weight w to walks and sets of walks on $\mathcal{N}$ by the following rules (see Fig. 2):

- Let $\sigma_v = (v)$ be a null walk in the node $v \in \mathcal{V}$; then $w(\sigma_v) = 1$.
- Let $\sigma = (u_0, u_1, u_2, \ldots, u_{p-1}, u_p)$ be a walk of length $p \geq 1$ on $\mathcal{N}$; then

$$w(\sigma) = \bigodot_{i=1}^{k} w(u_{i-1}, u_i).$$

- For empty set of walks $\varnothing$ it holds $w(\varnothing) = 0$.
- Let $S = \{\sigma_1, \sigma_2, \ldots\}$ be a set of walks in $\mathcal{N}$; then

$$w(\mathcal{S}) = \bigoplus_{\sigma \in \mathcal{S}} w(\sigma).$$



Semirings and Matrix Analysis of Networks, Fig. 2 Semiring operations and values of walks

Let σ_1 and σ_2 be compatible walks on $\mathcal{N}$: the end node of the walk σ_1 is equal to the start node of the walk σ_2. Such walks can be concatenated in a new walk $\sigma_1 \circ \sigma_2$ for which holds

$$w(\sigma_1 \circ \sigma_2) = \begin{cases} w(\sigma_1) \odot w(\sigma_2) & \sigma_1 \text{ and } \sigma_2 \text{ are compatible} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be finite sets of walks; then

$$w(\mathcal{S}_1 \cup \mathcal{S}_2) \oplus w(\mathcal{S}_1 \cap \mathcal{S}_2) = w(\mathcal{S}_1) \oplus w(\mathcal{S}_2).$$

In the special case when $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, it holds $w(\mathcal{S}_1 \cup \mathcal{S}_2) = w(\mathcal{S}_1) \oplus w(\mathcal{S}_2)$. Also the concatenation of walks can be generalized to sets of walks:

$$\mathcal{S}_1 \circ \mathcal{S}_2 = \{\sigma_1 \circ \sigma_2 : \sigma_1 \in \mathcal{S}_1,\ \sigma_2 \in \mathcal{S}_2,\ \sigma_1 \text{ and } \sigma_2 \text{ are compatible}\}.$$

It also holds $\mathcal{S} \circ \emptyset = \emptyset \circ \mathcal{S} = \emptyset$.

We denote by:

- $\mathcal{S}^k_{uv}$ the set of all walks of length k from node u to node v
- $\mathcal{S}^{(k)}_{uv}$ the set of all walks of length at most k from node u to node v
- $\mathcal{S}^*_{uv}$ the set of all walks from node u to node v
- $\overline{\mathcal{S}}_{uv}$ the set of all nontrivial walks from node u to node v
- $\mathcal{E}_{uv}$ the set of all simple walks (paths) from node u to node v

The following relations hold among these sets:

$$\mathcal{S}^k_{uv} \subseteq \mathcal{S}^{(k)}_{uv} \subseteq \mathcal{S}^*_{uv}$$
$$k \neq l \Leftrightarrow \mathcal{S}^k_{uv} \cap \mathcal{S}^l_{uv} = \emptyset$$
$$\mathcal{S}^{(k)}_{uv} = \bigcup_{i=0}^{k} \mathcal{S}^i_{uv}, \quad \mathcal{S}^*_{uv} = \bigcup_{k=0}^{\infty} \mathcal{S}^k_{uv}$$
$$k \geq n-1 : \mathcal{E}_{uv} \subseteq \mathcal{S}^{(k)}_{uv}$$
$$w\left(\mathcal{S}^{(k)}_{uv}\right) = \sum_{i=0}^{k} w\left(\mathcal{S}^i_{uv}\right).$$

A set of walks $\mathcal{S}$ is *uniquely factorizable* to sets of walks $\mathcal{S}_1$ and $\mathcal{S}_2$ if $\mathcal{S} = \mathcal{S}_1 \circ \mathcal{S}_2$, and for all walks $\sigma_1, \sigma'_1 \in \mathcal{S}_1$, $\sigma_2, \sigma'_2 \in \mathcal{S}_2, \sigma_1 \neq \sigma'_1, \sigma_2 \neq \sigma'_2$, it holds $\sigma_1 \circ \sigma_2 \neq \sigma'_1 \circ \sigma'_2$.

For example, for s, $0 < s < k$, a nonempty set $\mathcal{S}^k_{uv}$ is uniquely factorizable to sets $\mathcal{S}^s_{u\bullet}$ and $\mathcal{S}^{k-s}_{\bullet v}$, where $\mathcal{S}^s_{u\bullet} = \cup_{t \in \mathcal{V}} \mathcal{S}^s_{ut}$, etc.

Theorem 1 *Let the finite set $\mathcal{S}$ be uniquely factorizable to $\mathcal{S}_1$ and $\mathcal{S}_2$ or a semiring is idempotent. Then it holds*

$$w(\mathcal{S}_1 \circ \mathcal{S}_2) = w(\mathcal{S}_1) \odot w(\mathcal{S}_2).$$

The k-th power $\mathbf{W}^k$ of a square matrix $\mathbf{W}$ over $\mathbb{K}$ is unique because of associativity.

Theorem 2 *The entry w^k_{uv} of k-th power $\mathbf{W}^k$ of a value matrix $\mathbf{W}$ is equal to the value of all walks of length k from node u to node v:*

$$w\left(\mathcal{S}^k_{uv}\right) = \mathbf{W}^k[u,v] = w^k_{uv}.$$

Therefore if a network $\mathcal{N}$ is acyclic, then it holds for a value matrix $\mathbf{W}$:

$$\exists k_0 : \forall k > k_0 : \mathbf{W}^k = 0,$$

where k_0 is the length of the longest walk in the network.

If $\mathbf{W}$ is the network adjacency matrix over the combinatorial semiring, the entry w^k_{uv} counts the number of different walks of length k from u to v.

Let us denote

$$\mathbf{W}^{(k)} = \bigoplus_{i=0}^{k} \mathbf{W}^i.$$

In an idempotent semiring, it holds $\mathbf{W}^{(k)} = (1 \oplus \mathbf{W})^k$.

Theorem 3

$$w\left(\mathcal{S}^k_{uv}\right) = \mathbf{W}^{(k)}[u,v] = w^{(k)}_{uv}.$$

For the combinatorial semiring and the network adjacency matrix $\mathbf{W}$, the entry $w^{(k)}_{uv}$ counts the number of different walks of length at most k from u to v.

The matrix semiring over a complete semiring is also complete and therefore closed for $\mathbf{W}^* = \oplus_{k=0}^{\infty}\mathbf{W}^k$.

Theorem 4 *For a value matrix* $\mathbf{W}$ *over a complete semiring with closure* $\mathbf{W}^*$ *and strict closure* $\overline{\mathbf{W}}$ *hold:*

$$w\left(\mathcal{S}_{uv}^*\right) = \mathbf{W}^*[u,v] = w_{uv}^* \quad \text{and}$$
$$w\left(\overline{\mathcal{S}}_{uv}\right) = \overline{\mathbf{W}}[u,v] = \overline{w}_{uv}.$$

For the reachability semiring and the network adjacency matrix $\mathbf{W}$, the matrix $\overline{\mathbf{W}}$ is its transitive closure.

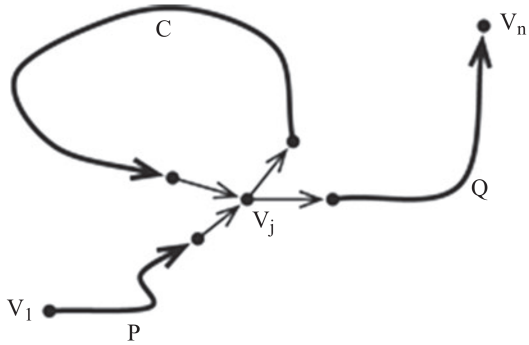
For the shortest paths semiring and the network value matrix $\mathbf{W}$, the entry w_{uv}^* is the value of the shortest path from u to v.

The paper (Quirin et al. 2008) could be essentially reduced to the observation that the structure $\left(\overline{\mathbb{R}}_0^+, \min, \boxed{r}, \infty, 0\right)$ is a (Pathfinder) complete semiring.

Let $(\mathbb{K}, \oplus, \odot, 0, 1)$ be an absorptive semiring and σ be a nonsimple walk from a set $\mathcal{S}_{uv}^*$. Therefore at least one node v_j appears more than once in σ. The part of a walk between its first and last appearance is a closed walk C (see Fig. 3). The whole walk can be written as $\sigma = P \circ C \circ Q$ where P is the initial segment of σ from u to the first appearance of v_j, and Q is the terminal segment of σ from the last appearance of v_j to v. Note that $P \circ Q$ is also a walk. The value of both walks together is $w(\{P \circ Q, P \circ C \circ Q\}) = w(P \circ Q)$. We see that the walks that are not paths do not contribute to the value of walks. Therefore $w\left(\mathcal{S}_{uv}^*\right) = w\left(\mathcal{E}_{uv}^*\right)$. This equality holds also for $\mathcal{S}_{uv}^* = \emptyset$.

Since the node set $\mathcal{V}$ is finite, also the set $\mathcal{E}_{uv}$ is finite which allows us to compute the value $w\left(\mathcal{S}_{uv}^*\right)$. We already know that $\mathbf{W}^* = \mathbf{W}^{(k)} = (\mathbf{1} \oplus \mathbf{W})^k$ for k large enough.

To compute the closure matrix $\mathbf{W}^*$ of a given matrix over a complete semiring $(\mathbb{K}, \oplus, \odot, 0, 1)$, we can use the Fletcher's algorithm (Fletcher 1980):



Semirings and Matrix Analysis of Networks, Fig. 3 Example of a walk that is not a path

$\mathbf{C}_0 = \mathbf{W}$
for $k = 1, \ldots, n$ **do**
  for $i = 1, \ldots, n$ **do**
    for $j = 1, \ldots, n$ **do**
      $c_k[i,j] = c_{k-1}[i,j] \oplus c_{k-1}[i,k] \odot (c_{k-1}[k,k])^* \odot c_{k-1}[k,j]$
  $c_k[k,k] = 1 \oplus c_k[k,k]$
$\mathbf{W}^* = \mathbf{C}_n$

If we delete the statement $c_k[k,k] = 1 \oplus c_k[k,k]$, we obtain the algorithm for computing the strict closure $\overline{\mathbf{W}}$. If the addition $\oplus$ is idempotent, we can compute the closure matrix in place – we omit the subscripts in matrices $\mathbf{C}_k$.

The Fletcher's algorithm is a generalization of a sequence of algorithms (Kleene, Warshall, Floyd, Roy) for computing closures on specific semirings.

Multiplication of Matrix and Vector

Let e_i be a unit vector of length n – the only nonzero element is at the i-th position and it is equal to 1. It is essentially a $1 \times n$ matrix. The product of a unit vector and a value matrix of a network can be used to calculate the values of walks from a node i to all the other nodes.

Let us denote $q_1^T = e_i^T \odot \mathbf{W}$. The values of elements of the vector q_1 are equal to the values of walks of the length 1 from a node i to all other nodes: $q_1[j] = w\left(\mathcal{S}_{ij}^1\right)$. We can calculate iteratively the values of all walks of the length s, $s = 2, 3, \ldots, k$ that start in the node

$$i: q_s^T = q_{s-1}^T \odot \mathbf{W}$$
$$\text{or } q_s^T = e_i^T \odot \mathbf{W}^s \text{ and } q_s[j] = w\left(\mathcal{S}_{ij}^s\right).$$

Similarly we get

$$q^{(k)^T} = e_i^T \odot \mathbf{W}^{(k)}, q^{(k)}[j] = w\left(\mathcal{S}_{ij}^{(k)}\right)$$
$$\text{and } q^{*^T} = e_i^T \odot \mathbf{W}^*, q^*[j] = w\left(S_{ij}^*\right).$$

This can be generalized as follows. Let $\mathcal{I} \subseteq \mathcal{V}$ and $e_{\mathcal{I}}$ is the characteristic vector of the set $\mathcal{I}$ – it has value 1 for elements of $\mathcal{I}$ and is 0 elsewhere. Then, for example, for $q_k^T = e_{\mathcal{I}}^T \odot \mathbf{W}^k$, it holds $q_k[j] = w\left(\cup_{i \in \mathcal{I}} \mathcal{S}_{ij}^k\right)$.

Future Directions

New network analysis problems are emerging all the time. For some of them a semiring-based approach can prove to be useful. Recently we proposed a longitudinal approach to analysis of temporal networks based on semirings of temporal quantities (Batagelj and Praprotnik 2016).

Cross-References

- ► Eigenvalues, Singular Value Decomposition
- ► Iterative Methods for Eigenvalues/Eigenvectors
- ► Markov Chain Monte Carlo Model
- ► Matrix Algebra, Basics of
- ► Spectral Analysis

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