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Semirings for temporal network analysis

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In the article we describe a new algebraic approach to the temporal network analysis based on the notion of temporal quantities. We define the semiring for computing the foremost journey and the traveling semirings for the analysis of temporal networks where the latency is given, the waiting times are arbitrary, and some other information on the links are known. We use the operations in the traveling semiring to compute a generalized temporal betweenness centrality of the vertices that corresponds to the importance of the vertices with respect to the ubiquitous foremost journeys in a temporal network.

Keywords: Temporal quantity, temporal network, latency, semiring, centrality measure, betweenness.

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1. Introduction

Network analysis is used for different purposes in operations research, social sciences and many other scientific fields. A lot of research is done in communication networks, logistics, and the internet. The interest in network analysis increased in recent times, mostly due to the availability of big data and the global interest in data analysis. The growth of the internet and the amount od information available gave rise to many methods for the analysis of big data and sparse networks. In the last decades, especially temporal networks are of interest where the time dimension is also considered.

In a temporal network, the presence and the activity of vertices and links can change through time. Temporal data was added to networks in different scientific fields, for example transport systems Bell & Iida (1997); Correa & Stier-Moses (2011) and project management (CPM, Pert) in operations research Moder & Phillips (1970). An overview of temporal network analysis is given in Holme & Saramäki (2012, 2013).

A lot of research is still confused with the terminology and the terms used in communication network analysis, transport networks, computer networks, etc. that are similar or even the same, define the same phenomenon with different notation and different words. For example temporal distance Xuan et al. (2003), reachability time Holme (2005), latency of the information and other terms name the same thing in different areas. The same thing happens with journeys Xuan et al. (2003) that are named temporal paths, time respecting paths or paths with schedules by other authors.

There is no established formal description of temporal networks. The common point of all current research is the time component and that the changes of the network are one of the key information about the network.

The beginnings of temporal network analysis are based on time slices of the network Santoro et al. (2011). The temporal network is represented as a sequence of static networks, representing the state of the temporal network at a chosen time point (interval).

Two different approaches aim to unify temporal networks theory in a way that could be used for all the different uses. One is the time-aggregated graph from George & Kim (2013). The other is the time-varying graph from Casteigts et al. (2012).

We feel that both descriptions lack the possibility of adding arbitrary information to the network vertices or links. They are both describing the presence with explicit functions which also seems too complex. In Batagelj & Praprotnik (2014) and Praprotnik & Batagelj (2015) we proposed a new way for the temporal network description which remedies both of these shortcomings.

In the article, we shortly explain our description of temporal networks and study the case of temporal networks that is an extension of static networks and of temporal networks with zero latency and zero waiting times described in our previous articles Batagelj & Praprotnik (2014); Praprotnik & Batagelj (2015). We define a mathematical model for the description of temporal networks that allows for the presence / activity of the vertices / links and for the vertex properties and the link weights to change through time. The amount of the information that can be described with our representation of temporal networks is not limited. We construct semirings with operations that allow us to define and compute a simple vertex centrality measure in a temporal network.

Most of the static network analysis based on paths has been difficult to generalize to the case of temporal network because of the obvious differences – in static networks the shortest path always includes the shortest subpaths which is not true in temporal networks (we address this issue in more detail at the end of the article). Also, these measures cannot be generalized for the time slices approach as the temporal network can be disconnected at every time point and connected through time (think of the network of e-mail messages). The analysis of path based indices has to be done on dynamic networks that include the latency information.

For some special cases, there were steps taken to compute shortest, fastest and foremost journeys Xuan et al. (2003). But the complexity of the standard problems of network analyisis can be a lot greater in temporal networks. For example, the problem of strongly connected components in temporal networks is NP complete Bhadra & Ferreira (2003); Nicosia et al. (2011).

With this article, we make a step towards unifying temporal networks description and to adding information to the vertices and links of the temporal network. We also provide a way to combine different information in a useful way. One such example is the generalization of the betweenness centrality.

In Section 2 we present some basic definitions and notation used in the rest of the paper.

In Section 3 we define semirings and describe their use in network analysis. We give some examples that we need for the description and better understanding of temporal semirings.

In Section 4 we present the definitions of our new approach to the temporal network analysis. We introduce the notion of temporal quantities and the temporal semirings for the analysis of temporal networks with zero latency and zero waiting time. We introduce the semiring of increasing functions and explain how it is used in computing the foremost journeys – we get the first arrival semiring. The traveling semirings take into account additional network information, besides the latency.

In Section 5 we explain a possible use of the traveling semiring – a generalization of the betweenness centrality.

We conclude with directions for future work in Section 6. Our work opens a lot of different future research possibilities.

2. Definitions and notation

Definition 1 A *graph* $\mathcal{G}$ is an ordered pair of sets $(\mathcal{V},\mathcal{L})$, the set $\mathcal{V}$ is the set of *vertices* and the set $\mathcal{L}$ is the set of links between vertices. The links between the vertices u and v can be *directed (arcs)* (u,v) or *undirected (edges)* $\{u,v\}$. With $\ell(u,v)$ we tell that the link ℓ goes from u to v. If for an arc ℓ it holds $\ell(u,v)$ we say that ℓ starts at u and ends at v.

With n we denote the number of vertices $|\mathcal{V}|$ and with m the number of links $|\mathcal{L}|$. We assume that n and m are finite.

Definition 2 A *network* $\mathcal{N} = (\mathcal{V},\mathcal{L},\mathcal{P},\mathcal{W})$ consists of the graph $\mathcal{G} = (\mathcal{V},\mathcal{L})$ with additional information about the values (*weights*) of links $\mathcal{W}$ and the values (*properties*) of the vertices $\mathcal{P}$.

Definition 3 A *walk* in a graph $\mathcal{G}$ with a *start* at the vertex v_0 and an *end* at the vertex v_p is a finite alternating sequence of vertices and links

$$\pi = v_0\ell_1 v_1\ell_2 v_2 \dots \ell_p v_p$$

iff $\ell_i(v_{i-1},v_i)$, $i = 1,2,\dots,p$. The *length* of a walk is the number p of links it contains. The sequence π is a *semiwalk* iff the direction of the links is not important, that is $\ell_i(v_{i-1},v_i)$ or $\ell_i(v_i,v_{i-1})$ for all $i = 1,2,\dots,p$. A walk is *closed* iff it starts and ends at the same vertex, $v_0 = v_p$. A walk without repeating vertices is an *elementary walk* or a *path*.

Definition 4 A *value matrix* $\mathbf{A}$ of a network $\mathcal{N} = (\mathcal{V},\mathcal{L},w)$ is defined as

$$\mathbf{A} = [a_{uv}]_{u,v\in\mathcal{V}} = \begin{cases} w(u,v), & (u,v) \in \mathcal{L}, \\ 0, & \text{otherwise.} \end{cases}$$

In our notation $0 \in \mathbb{N}$. We denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and $\overline{\mathbb{R}_0^+} = \mathbb{R}_0^+ \cup \{\infty\}$.

3. Semirings

Semirings are frequently used in network analysis Baras & Theodorakopoulos (2010); Carre (1979); Dolan (2013); Gondran & Minoux (2008); Mohri (2002); Zimmerman (1981). In this section, we describe semirings that are used most frequently and are later generalized for the analysis of temporal networks.

Definition 5 Let $a,b,c \in A$. The set A with binary operations addition $\oplus$ and multiplication $\odot$, neutral element 0 and unit 1, denoted with $A(\oplus,\odot,0,1)$, is a *semiring*, when the following conditions hold:

- the set A is a commutative monoid for the addition $\oplus$ with a neutral element 0 (the addition is commutative, associative and $a \oplus 0 = a$ for all $a \in A$);
- the set A is a monoid for the multiplication $\odot$ with the unit 1 (the multiplication is associative and $a \odot 1 = 1 \odot a = a$ for all $a \in A$);
- the addition distributes over the multiplication

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c) \quad \text{and} \quad (a \oplus b) \odot c = (a \odot c) \oplus (b \odot c);$$

- the element 0 is an absorbing element or zero for the multiplication

$$a \odot 0 = 0 \odot a = 0 \text{ for all } a \in A.$$

In all cases we assume precedence of the multiplication over the addition. The last point in the definition of semirings is omitted by some authors. We need it in order to construct a matrix semiring over the semiring A. If all the points in the definition, except for the last one, hold for a given set A, it can be extended with the element ⌘, for which by definition

$$a \oplus ⌘ = ⌘ \oplus a = a \quad \text{and} \quad a \odot ⌘ = ⌘ \odot a = ⌘$$

holds for all $a \in A \cup \{⌘\}$. In the extended set $A_⌘ = A \cup \{⌘\}$ the element ⌘ is a zero by the definition and $(A_⌘, \oplus, \odot, ⌘, 1)$ is a semiring.

Definition 6 A semiring is *complete* iff the addition is well defined for countable sets and the distributivity laws still hold.

Definition 7 The addition is *idempotent* iff $a \oplus a = a$ for all $a \in A$.

Definition 8 A complete semiring $(A, \oplus, \odot, 0, 1)$ is *closed* iff an additional unary operation *closure* $\star$ is defined in it and

$$a^\star = 1 \oplus (a \odot a^\star) = 1 \oplus (a^\star \odot a) \text{ for all } a \in A.$$

We define a *strict closure* $\overline{a}$ in a closed semiring as

$$\overline{a} = a \odot a^\star.$$

There can be different closures in the same semiring. A complete semiring is closed when the closure is defined with

$$a^\star = \bigoplus_{k \geq 0} a^k. \tag{3.1}$$

In the rest of the article the term closure describes the operation from the equation (3.1).

Definition 9 A semiring $(A, \oplus, \odot, 0, 1)$ is *absorptive* iff for every $a, b, c \in A$ it holds

$$(a \odot b) \oplus (a \odot c \odot b) = a \odot b.$$

Because of the distributivity and the existence of the unit, it is enough to check that $1 \oplus c = 1$ for every $c \in A$ for the validity of the absorption law. In absorptive semirings also $a^\star = 1$ for all $a \in A$. An absorptive semiring is idempotent.

Definition 10 Over the semiring $(A, \oplus, \odot, 0, 1)$ we construct the *semiring of square matrices* $A^{n\times n}$ of order n which consist of the elements from A. The addition and the multiplication in the matrix semiring are defined in the usual way:

$$(\mathbf{A} \oplus \mathbf{B})_{ij} = a_{ij} \oplus b_{ij} \quad \text{and} \quad (\mathbf{A} \odot \mathbf{B})_{ij} = \bigoplus_{k=1}^{n} a_{ik} \odot b_{kj}, \quad i, j = 1, 2, \ldots, n.$$

Note that the operations on the left hand side operate in the matrix semiring $A^{n\times n}$ and the operations on the right hand side operate in the underlying semiring A.

For computing the closure $\mathbf{A}^\star$ of the network value matrix $\mathbf{A}$ over a complete semiring $(A, \oplus, \odot, 0, 1)$ the Fletcher's algorithm can be used. It is described in Fletcher (1980).

3.1 *The use of semirings in network analysis*

In network analysis semirings are used to combine weights on the links of the network. Combining the weights, we can observe different network properties. There are two basic cases – combining weights of two parallel links between two vertices or the weights of two sequential links between three vertices. The weights on the parallel links are combined using the semiring addition and the weights on the sequential links are combined using the semiring multiplication. A graphical representation is given in Figure 1. Using the semiring operations, the weights of links can be extended to walks and to sets of walks in the network Batagelj (1994).

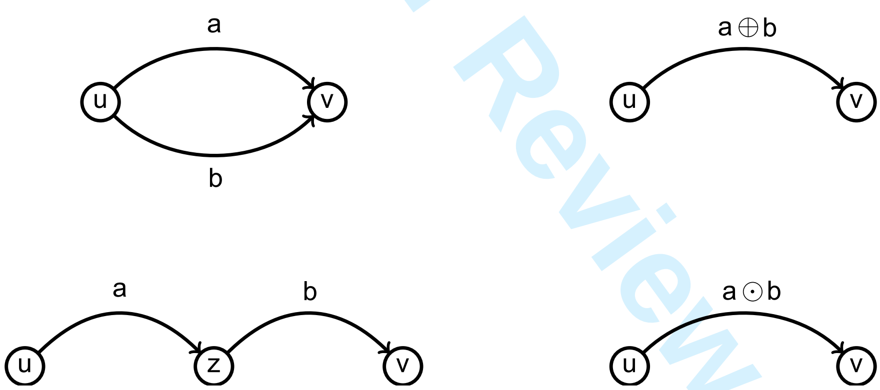

Figure 1: The semiring addition and the semiring multiplication in networks.

3.1.1 *Combinatorial semiring*

The combinatorial semiring is the semiring of the natural numbers for the usual addition and multiplication $(\overline{\mathbb{N}}, +, \cdot, 0, 1)$. In some cases other number sets are used, for example $\overline{\mathbb{R}_0^+}$. This semiring is complete and closed for $a^\star = \sum_{k\geq 0} a^k$. It is not absorptive and the addition is not idempotent.

In network analysis, the combinatorial semiring is used when the weights of links represent the number of ways to traverse them. The semiring addition and multiplication correspond to the rule of sum and the rule of product used in combinatorics Riordan (1958).

3.1.2 *Shortest paths semiring* The shortest paths semiring is defined as $(\overline{\mathbb{R}_0^+}, \min, +, \infty, 0)$. It is complete, commutative (also the semiring multiplication is commutative), and absorptive. It is closed and $a^\star = \min\{0, a + a^\star\} = 0$ for all $a \in \overline{\mathbb{R}_0^+}$. If the set $\overline{\mathbb{N}}$ is used instead of $\overline{\mathbb{R}_0^+}$, the semiring is called *tropical*.

The shortest paths semiring is used in the classical shortest paths problem:

A network $\mathcal{N} = (\mathcal{V}, \mathcal{L}, w)$ with weights on links $w : \mathcal{L} \to \overline{\mathbb{R}_0^+}$ and a (source) vertex $s \in \mathcal{V}$ are given. The value $w(u,v)$ represents the length of the link from u to v. We would like to compute all lengths of the shortest paths from s to other vertices $v \in \mathcal{V} \setminus \{s\}$. The usual solution is using dynamic programming: Define $d(s) = 0$ and compute the distances to other vertices $v \in \mathcal{V} \setminus \{s\}$ using Bellman's equation

$$d(v) = \min_{u \in \mathcal{V}} \{d(u) + w(u,v)\}. \tag{3.2}$$

3.1.3 *Geodetic semiring* In a set $A = \overline{\mathbb{R}_0^+} \times \overline{\mathbb{N}}$ the addition

$$(a,i) \oplus (b,j) = \left(\min(a,b), \begin{cases} i, & a < b \\ i + j, & a = b \\ j, & a > b \end{cases} \right)$$

and the multiplication

$$(a,i) \odot (b,j) = (a + b, i \cdot j)$$

are defined. For these operations $\big(A, \oplus, \odot, (\infty, 0), (0,1)\big)$ is a complete closed semiring Batagelj (1994) for the closure

$$(a,i)^\star = \begin{cases} (0,\infty), & a = 0, i \neq 0, \\ (0,1), & \text{otherwise.} \end{cases}$$

It is called a geodetic semiring. It is not idempotent.

The geodetic semiring is a combination of the shortest paths semiring and the combinatorial semiring. It is used to compute the length and the number of the shortest paths between pairs of vertices.

4. Semirings for temporal networks

Definition 11 A *temporal network* $\mathcal{N} = (\mathcal{V}, \mathcal{L}, \mathcal{T}, \mathcal{P}, \mathcal{W})$ is an ordinary (static) network $(\mathcal{V}, \mathcal{L}, \mathcal{P}, \mathcal{W})$ with an added time dimension $\mathcal{T}$. The set $\mathcal{T}$ of time points $t \in \mathcal{T}$ is a lifetime of the network. The lifetime $\mathcal{T}$ is usually a subset of integers $\mathcal{T} \subseteq \overline{\mathbb{Z}}$ or a subset of reals $\mathcal{T} \subseteq \overline{\mathbb{R}}$. In general, a linearly ordered set is sufficient. In the following we use $\mathcal{T}$ as a semiring with operations $\oplus = \min$ and $\odot = +$.

For the operations on temporal networks with zero latency, described in our articles Batagelj & Praprotnik (2014); Praprotnik & Batagelj (2015), we assumed $\mathcal{T} \subseteq \overline{\mathbb{N}}$.

In a temporal network the vertices $v \in \mathcal{V}$ and the links $\ell \in \mathcal{L}$ are not necessarily present or active all the time. Let $T(v)$, $T \in \mathcal{P}$, be the set of time points in which the vertex v is present; and let $T(\ell)$, $T \in \mathcal{W}$, be the set of time points in which the link ℓ is active. We require that the following *consistency condition* holds: If a link $\ell(u,v)$ is active at the time t its end vertices u and v must be present at the time t. Formally,

$$T(\ell(u,v)) \subseteq T(u) \cap T(v). \tag{4.1}$$

Definition 12 The static network consisting of links and vertices present in a temporal network at the time $t \in \mathcal{T}$ is denoted with $\mathcal{N}(t)$ and is called a *time slice* of the temporal network at the time t.

Let $\mathcal{T}' \subset \mathcal{T}$. Time slices are generalized to the set $\mathcal{T}'$ as

$$\mathcal{N}(\mathcal{T}') = \bigcup_{t \in \mathcal{T}'} \mathcal{N}(t).$$

When we are interested in walks in temporal networks, there are usually additional information on the links of the network.

Definition 13 The *latency* $\tau \in \mathcal{W}$, $\tau : \mathcal{L} \times \mathcal{T} \to \overline{\mathbb{R}_0^+}$. The value of $\tau(\ell, t)$ represents the time needed to traverse the link ℓ if the transition is started at the time t. If the latency τ is omitted, we assume $\tau(\ell, t) = 0$ for all $\ell \in \mathcal{L}$ and for all $t \in \mathcal{T}$.

Definition 14 The *weight* $w \in \mathcal{W}$, $w : \mathcal{L} \times \mathcal{T} \to \overline{\mathbb{R}}$, with values $w(\ell, t)$ representing length, cost, flow, etc. on the link ℓ if the transition is started at the time t. If the weight w is omitted, we assume $w(\ell) = 1$ for all links $\ell \in \mathcal{L}$ and all times $t \in \mathcal{T}$. In some cases the weights are structured.

Definition 15 A walk in a temporal network is called a *journey*. The journey $\sigma(v_0, v_k, t_0)$ from the start vertex v_0 to the end vertex v_k with the begining t_0 is a finite sequence

$$(t_0, v_0, (t_1, \ell_1), v_1, (t_2, \ell_2), v_2, \dots, v_{k-2}, (t_{k-1}, \ell_{k-1}), v_{k-1}, (t_k, \ell_k), v_k),$$

where $v_i \in \mathcal{V}, i = 0, 1, \dots, k$, and $\ell_i \in \mathcal{L}, t_i \in \mathcal{T}, i = 1, 2, \dots, k$. The links have to link the appropriate vertices, $\ell_i(v_{i-1}, v_i)$.

We denote $t'_0 = t_0$, $t'_i = t_i + \tau(\ell_i)$, $i = 1, 2, \dots, k$. For a journey $t'_{i-1} \le t_i$ has to hold and the link ℓ_i has to be present in the time interval $[t_i, t'_i]$ for all $i = 1, 2, \dots, k$. Also the vertex v_0 has to be present at the time t_0.

The triples $v_{i-1}, (t_i, \ell_i)$ in the definition of journeys tell that we started from the vertex v_{i-1} at the time t_i along the link ℓ_i.

Note that by the consistency condition it also holds that the vertex v_i is present at the time t_i and the vertex v_{i+1} is present at the time t'_i.

Definition 16 A journey is *regular* if the vertex v_i is present while waiting in the vertex for the next transition, that is during the time interval $[t'_{i-1}, t_i]$, $i = 1, 2, \dots, k-1$.

Definition 17 A journey σ has a (graph) *length* equal to the number of included links k, $|\sigma| = k$. The *duration* of the journey is equal to $t(\sigma) = t'_k - t_0$ and the *value* of the journey is equal to

$$w(\sigma) = w(\ell_1, t_1) \odot w(\ell_2, t_2) \odot \cdots \odot w(\ell_k, t_k) = \bigodot_{(t, \ell) \in \sigma} w(\ell, t)$$

for the multiplication in the appropriate semiring.

Definition 18 The time t_0 is the *begining* of the journey, the time t_1 is the *departure* and t'_k is the *arrival* (end of the journey). The time $t'_k - t_1$ is called a *strict duration* of the journey. Times $t_i - t'_{i-1}$ are the *waiting times* of the journey.

Definition 19 A *jump* is a journey inside a given network time slice $\mathcal{N}(t)$. Jumps have zero latency and zero waiting times.

Definition 20 The *fastest* journey is the one with the smallest strict duration. The *foremost* journey is the one with the smallest arrival time. The *cheapest* journey is the one with the smallest value.

Definition 21 A part of the journey $\sigma(v_0, v_k, t_0)$ from the vertex v_i to the vertex v_j with the beginning at the time t_i,

$$(t_i, v_i, (t_{i+1}, \ell_{i+1}), v_{i+1}, \ldots, v_{j-1}, (t_j, \ell_j), v_j),$$

is caled a *stage* of the journey. An *ubiquitous foremost journey* is the foremost journey for which every stage is a foremost journey between the vertices v_i and v_j with the beginning t_i.

It has been shown in Xuan et al. (2003) that if there exists a journey between two vertices, then the ubiquitous foremost journey exists between them.

4.1 *Temporal quantities*

In temporal networks besides the presence or absence of vertices and links, also the values of vertex and link properties change through time. For the description of the temporal properties we introduced *temporal quantities* in Batagelj & Praprotnik (2014). Let $a(t)$ be the value of the property a at the time t. We assume that the values $a(t)$ of the function a belong to the semiring $(A, \oplus, \odot, 0, 1)$. The vertex or the link that a is describing is not necessarily present at all times. Therefore the function a is not defined for all values $t \in \mathcal{T}$.

Definition 22 Let $(A, \oplus, \odot, 0, 1)$ be a semiring and let the function $a : T_a \to A$ describe a temporal property in a temporal network. A *temporal quantitiy* $\hat{a} : \mathcal{T} \to A$ is an extension of the function a,

$$\hat{a}(t) = \begin{cases} a(t), & t \in T_a, \\ 0, & t \in \mathcal{T} \setminus T_a. \end{cases}$$

Note that the values of temporal quantities while the vertex or the link is not present are defined as the zero of the semiring A. This means that the values along the sequential links are equal to 0 (describing nonexistence) if one of the sequential links does not exist.

In the rest of the article we denote temporal quantities with a instead of with $\hat{a}$.

4.2 *Temporal semirings*

In this section, the latency and the waiting times in the temporal network are equal to zero. We described the temporal semirings in more detail and provided algorithmic support in our articles Batagelj & Praprotnik (2014); Praprotnik & Batagelj (2015).

Definition 23 Let $A_{\mathcal{T}}$ be a set of all temporal quantities over the chosen semiring $(A, \oplus, \odot, 0, 1)$ for the lifetime $\mathcal{T}$, that is $A_{\mathcal{T}} = \{a : \mathcal{T} \to A\}$. In the set $A_{\mathcal{T}}$ we define the *addition*

$$(a \oplus b)(t) = a(t) \oplus b(t),$$

and the *multiplication*

$$(a \odot b)(t) = a(t) \odot b(t).$$

The operations on the left hand side operate in the set $A_{\mathcal{T}}$ of temporal quantities over the semiring A for the lifetime $\mathcal{T}$, and the operations on the right hand side operate in the semiring A.

Theorem 1 The set $A_{\mathcal{T}}$ for the operations from the definition 23 is a semiring with the zero $0(t) = 0$, $t \in \mathcal{T}$, and the unit $1(t) = 1$, $t \in \mathcal{T}$.

Proof. The operations are defined pointwise and the semiring properties in $A_{\mathcal{T}}$ follow from the properties of the semiring A. □

Definition 24 Let A be a combinatorial (shortest paths, geodetic, etc.) semiring. The semiring $A_{\mathcal{T}}$ is called a *temporal combinatorial (shortest paths, geodetic, etc.) semiring.*

We can construct a matrix semiring over the temporal semirings. Such matrices can be used to describe temporal networks. Because the values of $a(t)$ and $b(t)$ in the definition 23 correspond to the same time point t, the latency and the waiting times are restricted to zero for the whole lifetime. The use of this semiring in temporal networks is restricted to jumps and not to arbitrary journeys for the operations to make sense.

4.3 *Semiring of increasing functions*

Definition 25 A function f is *increasing* iff $f(x) \geq f(y)$ for all x, y of its domain for which $x \geq y$. We say that a function f is *expanding* if $f(x) \geq x$ for all x of its domain.

Theorem 2 The set

$$A = \left\{ \ f : \overline{\mathbb{N}} \to \overline{\mathbb{N}}; \text{ function } f \text{ is increasing and expanding } \right\}$$

is a semiring for the operations

$$f \oplus g = \min(f, g) \quad \text{ and } \quad f \odot g = g \circ f.$$

The zero is a function $f \equiv \infty$ and the unit is the identity function $f = id$. For the domain or codomain of functions f we could also choose the sets $\overline{\mathbb{R}_0^+}$, $\overline{\mathbb{Z}}$, or $\overline{\mathbb{R}}$.

Proof. This semiring is very similar to the semiring from (Gondran & Minoux, 2008, p. 346, Section 4.2.1). □

Definition 26 The semiring A from theorem 2 is called the *semiring of increasing functions.*

The semiring of increasing functions is complete, idempotent ($\min(f, f) = f$), closed for $f^\star = 1 \oplus f \odot f^\star = \min(id, f^\star \circ f) = id$, and absorptive ($\min(id, f) = id$) because $f^\star$ and f are increasing and expanding functions.

4.4 *First arrival semiring*

We start with an equation, similar to Bellman's equation (3.2), for the solution of finding the foremost journeys in a temporal network.

Let a temporal quantity a_{uv} describe latency along the link (u, v) and let $T(u, v, t_0)$ be the first possible time at which we can arrive at the vertex v if we start at the vertex u at the time t_0. Then

$$T(u, u, t_0) = t_0$$

and

$$T(u, v, t_0) = \min_{w:(w,v) \in \mathcal{L}} \left(\min_{t \geq T(u,w,t_0)} (t + a_{wv}(t)) \right). \tag{4.2}$$

If we are interested in the duration, we subtract the begining t_0 from the result.

We would like to construct a semiring that gives us this equation, similarly to the way that the shortest paths semiring gives Bellman's equation. The semiring operations are not obvious, as there are three operations (two minimums and the addition) in equation 4.2.

What we can see is that it is useful to define a function (temporal quantity) that tells the first arrival time for the given start, end, and begining of the journey.

From the network interpretation we can see what the appropriate semiring addition and multiplication are:

Let our journey take two sequential links (u,w) and (w,v). The first arrival time at the vertex w along the link (u,w) is described with the temporal quantity f, and the first arrival at the vertex v along the link (w,v) is described with the temporal quantity g. The corresponding journey is outlined in Figure 2.

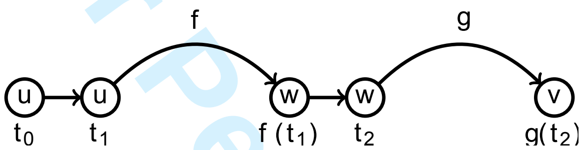

Figure 2: A journey along sequential links.

From the begining t_0 of the journey we wait in the vertex u for some favorable time t_1 when we move along the link (u,w). This part of the journey ends at the time $f(t_1)$. Afterwards we wait for a favorable time t_2 in the vertex w. At that time we move along the link (w,v). The journey ends at the time $g(t_2)$. We are interested in the first arrival at the vertex v if we start at the vertex u at the time t_0 and visit the vertex w inbetween. That gives us an appropriate semiring multiplication

$$(f \odot g)(t_0) = \min_{\substack{t_1 \geq t_0 \\ t_2 \geq f(t_1)}} g(t_2).$$

We note that if f and g are increasing functions, this equation is equivalent to

$$(f \odot g)(t_0) = g(f(t_0)) = (g \circ f)(t_0).$$

We also point out that the multiplication is not commutative which means that the order in which the links are traversed is important. That is in accordance with our intuition.

When the journey can take us along two parallel links (one possibility is presented in Figure 3) we start at the time t_0 and wait for the time t_1, when it pays to go along the edge for which the arrival times are described with the function g. This journey ends at the time $g(t_1)$. If we wish to take the other link, where the arrival times are described with the function f, we wait for some other time t_2 and arrive at v at the time $f(t_2)$. The first arrival time is the smallest of the times $f(t_2)$ and $g(t_1)$.

That is

$$(f \oplus g)(t_0) = \min_{\substack{t_1 \geq t_0 \\ t_2 \geq t_0}} \big(f(t_2), g(t_1)\big) = \min_{t \geq t_0} \big(f(t), g(t)\big).$$

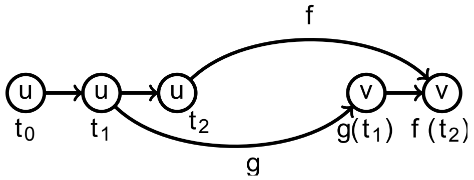

Figure 3: A journey on parallel links.

When f and g are increasing, the equation is equivalent to

$$(f \oplus g)(t_0) = \min(f(t_0), g(t_0)).$$

The two appropriate operations are exactly the ones from the semiring of increasing functions.

Let the values of the temporal quantity a represent the latency along the link. Remember that $a(t) = \infty$ at times $t \in \mathcal{T} \setminus T_a$. We assign a function f to the temporal quantity a:

$$a \mapsto f: \; f(t) = \min_{\tau \geq t}\{\tau + a(\tau)\}. \tag{4.3}$$

The function f is increasing and expanding if $a \geq 0$ which it usually is as the travel times are nonnegative. If a is describing the latency along the link (u,v) the function f is describing the first arrival time from u to v. The value $f(t)$ is the first arrival if we begin the journey at the time t.

The first arrival times in a temporal network with arbitrary waiting times and given latencies can be computed with the addition and multiplication in the semiring of increasing functions.

Definition 27 Let $\mathcal{N} = (\mathcal{V}, \mathcal{L}, \mathcal{T}, a)$ be a temporal network and let the temporal quantity $a : \mathcal{T} \to \mathcal{T}$ describe the latency. We assign a function f to the temporal quantity a as in the equation (4.3). The semiring

$$\mathbb{T} = \left(\{f : \mathcal{T} \to \mathcal{T}\}, \min, \circ, \infty, id\right)$$

is called the *first arrival semiring.*

4.5 *Generalized geodetic semirings*

The generalized geodetic semirings are defined in a very similar way as the geodetic semiring from Section 3.1.3.

Definition 28 In a set $\mathcal{T} \times A$, where $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is an arbitrary complete semiring (combinatorial, shortest paths, geodetic, etc.), the operations *addition* $\boxplus$ and *multiplication* $\boxtimes$ are defined as

$$(\tau, a) \boxplus (\sigma, b) = \left(\min(\tau, \sigma), \begin{cases} a, & \tau < \sigma, \\ a \oplus b, & \tau = \sigma, \\ b, & \tau > \sigma \end{cases}\right)$$

and

$$(\tau, a) \boxtimes (\sigma, b) = (\tau + \sigma, a \odot b).$$

Theorem 3 The set $\mathcal{T} \times A$ is a semiring for the addition $\boxplus$ and the multiplication $\boxtimes$. The zero is $(\infty, \mathbf{0})$ and the unit is $(0, \mathbf{1})$.

Proof. The construction is almost identical to the one for the geodetic semiring and the semiring properties follow in the same way as in Batagelj (1994) from the properties of the operations in $\mathcal{T}$ and A. □

Definition 29 The semiring $G_{\mathcal{T}\times A} = \left(\mathcal{T} \times A, \boxplus, \boxtimes, (\infty, \mathbf{0}), (0, \mathbf{1})\right)$ is called a *generalized geodetic semiring.*

4.6 *Traveling semirings*

The next question is how to combine different information on the links. For example, latency and the number of ways to traverse it or latency and distance.

Let the temporal quantity $a : \mathcal{T} \to \mathcal{T}$ describe the latency and let the temporal quantity $i \in A_{\mathcal{T}}$ over a chosen semiring $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ describe some other information about the link.

We want to compute

$$(f,n)(t) = \left(\min_{\tau \geq t} (a(\tau) + \tau), \bigoplus_{\substack{\sigma \in \mathrm{Argmin}_{\tau \geq t}(a(\tau)+\tau) \\ \sigma \geq t}} i(\sigma) \right).$$

The first component f stays the same as in the first arrival semiring (equation (4.3)) and tells the first arrival along the link after the time t. In the second component n we sum (over the chosen semiring A) the values along the links on which the minimal arrival time is achieved and that start after the time t.

First, we do a simple transformation

$$(a,i) \mapsto (a',i) \quad \text{where} \quad a'(t) = a(t) + t$$

from which we get

$$(f,n)(t) = \left(\min_{\tau \geq t} a'(\tau), \bigoplus_{\substack{\sigma \in \mathrm{Argmin}_{\tau \geq t} a'(\tau) \\ \sigma \geq t}} i(\sigma) \right). \tag{4.4}$$

The last equation is simplified by summing over the corresponding generalized geodetic semiring $G_{\mathcal{T}\times A}$. The equation (4.4) can be rewritten as

$$(f,n)(t) = \boxplus_{\tau \geq t} (a'(\tau), i(\tau)). \tag{4.5}$$

Note that $f \in \mathbb{T}$ and $n \in A_{\mathcal{T}}$.

4.6.1 *Operations in traveling semirings*

The transformation (4.5) of the temporal quantities a, representing latency, and i, representing some other information, returns a pair (f,n) belonging to the set

$$G_A(\mathcal{T}) = \{(f,n);\ f \in \mathbb{T},\ n \in A_{\mathcal{T}}\}.$$

Definition 30 On a set of function pairs $G_A(\mathcal{T})$ we define the *addition* $\oplus$ and the *multiplication* $\diamond$ with

$$\big((f,n) \oplus (g,m)\big)(t) = (f,n)(t) \boxplus (g,m)(t),$$
$$\big((f,n) \diamond (g,m)\big)(t) = \big((g\circ f)(t), n(t) \odot m(f(t))\big).$$

The operation $\boxplus$ is the addition in the generalized geodetic semiring $G_{\mathcal{T}\times A}$ and the operation $\odot$ is the multiplication in the semiring A.

The definitions can be read as: If there are two parallel links, we choose the one that arrives first and preserve the same additional value. If both parallel links arrive at the same time, we sum the corresponding additional values.

On sequential links the arrival time is the same as the arrival over the second link. The journey along the second link can begin after the first arrival along the first link (time $f(t)$). The value of the second component is the value on the first link if we start the journey after the time t multiplied by the value of the second link if we traverse the link after the time $f(t)$.

The first component tells the first arrival and the second component tells additional values for the ubiquitous foremost journey, depending on the semiring A. If A is a combinatorial semiring, the second component tells the number of the ubiquitous foremost journeys. If A is the shortest paths semiring, the second component tells the length of the cheapest among the ubiquitous formost journeys.

Theorem 4 The set $G_A(\mathcal{T})$ is a semiring for the operations from the definition 30. The zero is a pair of constant functions $(\infty, \mathbf{0})$. The unit is $(id, \mathbf{1})$. The second component of the unit is a constant function.

Proof. The associativity, commutativity and the neutral element follow from the properties of the generalized geodetic semiring.

First, we show that $(id, \mathbf{1})$ is the unit

$$\big((f,n) \diamond (id,\mathbf{1})\big)(t) = \big(f(t), n(t)\odot \mathbf{1}\big) = (f,n)(t),$$
$$\big((id,\mathbf{1}) \diamond (f,n)\big)(t) = \big(f(t), \mathbf{1}\odot n(t)\big) = (f,n)(t).$$

and that $(\infty, \mathbf{0})$ is the zero

$$\big((f,n) \diamond (\infty,\mathbf{0})\big)(t) = (\infty, n(t)\odot \mathbf{0}) = (\infty,\mathbf{0}),$$
$$\big((\infty,\mathbf{0}) \diamond (f,n)\big)(t) = \big(f(\infty), \mathbf{0}\odot n(\infty)\big) = (\infty,\mathbf{0}), \text{ because } f \text{ is expanding.}$$

Now check the multiplication associativity and the distributivity. First the associativity:

$$\begin{aligned}\big(\big((f,n)\diamond(g,m)\big)\diamond(h,r)\big)(t) &= \big((g\circ f)(t), n(t)\odot m(f(t))\big)\diamond(h(t),r(t))\\ &= \big((h\circ g\circ f)(t), n(t)\odot m(f(t))\odot r((g\circ f)(t))\big)\\ \big((f,n)\diamond\big((g,m)\diamond(h,r)\big)\big)(t) &= (f,n)(t)\diamond\big((h\circ g)(t), m(t)\odot r(g(t))\big)\\ &= \big((h\circ g\circ f)(t), n(t)\odot m(f(t))\odot r(g(f(t)))\big).\end{aligned}$$

We get the same result in both cases, therefore the associativity holds. Check for distributivity:

$$((h,r)\diamond(f,n))(t) = ((f\circ h)(t), r(t)\odot n(h(t))),$$
$$((h,r)\diamond(g,m))(t) = ((g\circ h)(t), r(t)\odot m(h(t)))$$

and

$$((h,r)\diamond(f,n)\oplus(h,r)\diamond(g,m))(t) =$$
$$\left(\min(f(h(t)),g(h(t))),\begin{cases} r(t)\odot n(h(t)), & f(h(t))<g(h(t))\\ r(t)\odot(n(h(t))\oplus m(h(t))), & f(h(t))=g(h(t))\\ r(t)\odot m(h(t)), & f(h(t))>g(h(t))\end{cases}\right).$$

We used the distributivity of the semiring A. The other side of the distributivity equation gives

$$((f,n)\oplus(g,m))(t) = \left(\min(f(t),g(t)),\begin{cases} n(t), & f(t)<g(t)\\ (n\oplus m)(t), & f(t)=g(t)\\ m(t), & f(t)>g(t)\end{cases}\right),$$

which we multiply from the left $(h,r)(t)\diamond$ and get

$$\left((\min(f,g)\circ h)(t), r(t)\odot\begin{cases} n(h(t)), & f(h(t))<g(h(t))\\ (n\oplus m)(h(t)), & f(h(t))=g(h(t))\\ m(h(t)), & f(h(t))>g(h(t))\end{cases}\right).$$

So the left distributivity holds. If we multiply $((f,n)\oplus(g,m))(t)$ on the right hand side $\diamond(h,r)(t)$ we get

$$\left((h\circ\min(f,g))(t),\begin{cases} n(t), & f(t)<g(t)\\ (n\oplus m)(t), & f(t)=g(t)\\ m(t), & f(t)>g(t)\end{cases}\odot r(\min(f(t),g(t)))\right),$$

which is the same as the results of the next computations

$$((f,n)\diamond(h,r))(t) = ((h\circ f)(t), n(t)\odot r(f(t))),$$
$$((g,m)\diamond(h,r))(t) = ((h\circ g)(t), m(t)\odot r(g(t))),$$

which adds with $\oplus$ to

$$\left(\min(h(f(t)),h(g(t))),\begin{cases} n(t)\odot r(f(t)), & h(f(t))<h(g(t)),\\ n(t)\odot r(f(t))\oplus m(t)\odot r(g(t)), & h(f(t))=h(g(t)),\\ m(t)\odot r(g(t)), & h(f(t))>h(g(t))\end{cases}\right).$$

The right distributivity holds, as f,g and h are increasing and the semiring A is distributive.

The distributivity holds and $G_A(\mathcal{T})$ is a semiring. □

Definition 31 Let A be a combinatorial (shortest paths, geodetic, etc.) semiring. The semiring

$$(G_A(\mathcal{T}),\oplus,\diamond,(\infty,\mathbf{0}),(id,\mathbf{1}))$$

is called the *traveling combinatorial (shortest paths, geodetic, etc.)* semiring.

5. Betweenness centrality

Determining important vertices in the network is one of the basic network analysis tools. A lot of different vertex centralities have been defined for static networks Wasserman & Faust (1994). One of the classical centralitity measures is the betweenness centrality Freeman (1977, 1978).

Definition 32 The *betweenness* of a vertex v in a network $\mathcal{N} = (\mathcal{V}, \mathcal{L}, \mathcal{W})$ is defined with

$$b(v) = \frac{1}{(n-1)(n-2)} \sum_{\substack{u,w \in \mathcal{V} \\ |\{v,u,w\}|=3}} \frac{n_{uw}(v)}{n_{uw}},$$

where n_{uw} is the number of the shortest paths from u to w and $n_{uw}(v)$ is the number of the shortest paths from u to w that include the vertex v. If $n_{uw} = 0$ we define $n_{uw}(v)/n_{uw} = 0$.

The betweenness centrality is based on the shortest paths in the network. The ratio $n_{uw}(v)/n_{uw}$ can be seen as the probability that the communication between u and w goes through v. Therefore, the betweenness centrality implicitly assumes that all the communication between the vertices of the network takes place only along the shortest paths. That is not necessarily the case and it is a known disadvantage of the betweenness centrality.

The betweenness centrality is motivated by network traffic monitoring. Which vertex has the most potential for influencing, security, connectivity, negotiations. It measures the strategic position of vertices.

In Batagelj & Praprotnik (2014) we described the generalization of betweenness centrality for temporal networks with zero latency. In this article, we aim to generalize it to networks with given latencies and arbitrary waiting times.

5.1 *Betweenness in temporal networks*

We will use the traveling combinatorial semiring $G_A(\mathcal{T})$ to define and compute the betweenness in temporal networks. In this semiring the pairs of temporal quantities (f,m) are viewed as the first arrival times, f, and as the number of possible traversals of links that result in the first arrival, m.

Definition 33 We define the *first arrival betweenness* with respect to the ubiquitous foremost journeys after the chosen time point t as

$$\mathbf{b}_v(t) = \frac{1}{(n-1)(n-2)} \sum_{\substack{u,w \in \mathcal{V} \\ |\{v,u,w\}|=3}} \frac{n_{uw(v)}(t)}{n_{uw}(t)}.$$

The $n_{uw}(t)$ denotes the number of ubiquitous foremost journeys from u to w that begin after the time t and the $n_{uw(v)}(t)$ denotes the number of ubiquitous foremost journeys from u to w that go through v and begin after the time t.

We point out that our definition has the same problem as the betweenness for static network. It assumes that all the communication / traffic in the temporal network travels along the ubiquitous foremost journeys.

In temporal networks it is not generally true, that the foremost journey includes only foremost stages which holds for shortest paths in static networks. See Figure 4 as an example. The weights on links are the latencies and the number of ways to cross them. The latency on the link (u,v) is 2 at the time point 1 and 3 at the time point 2. Between the vertices v and w the latency is equal to 2 at the time point 5. Outside the specified times the links are not present.

There are k foremost journeys between the vertices u and v that have the arrival time 3. Between the vertices v and w there are n foremost journeys. Between the vertices u and w

there are $(m+k)\cdot n$ foremost journeys. Our intuition does not distinguish between the waiting time in the vertex v and traveling along a link. The traveling semiring does. The link (u,v) with the weight $(3,m)$ is not taken into account in the semiring as it is not included among the ubiquitous foremost journeys between u and w.

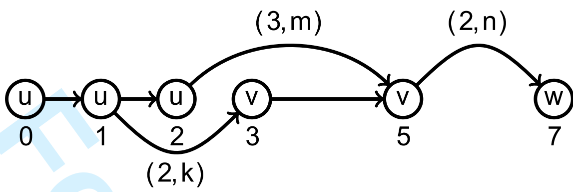

Figure 4: The foremost journey does not necessarily include only the foremost stages.

We compute the values $n_{uw}(t)$ and $n_{uw(v)}(t)$ from the closure $\mathbf{B}$ of a temporal network matrix over the traveling combinatorial semiring in a similar way as for the static case. The matrix $\mathbf{B}$ consists of pairs of temporal quantities $\big(f_{uv}(t), n_{uv}(t)\big)$. The value $f_{uv}(t)$ is the first arrival time for journeys from u to v with the begining t. The value $n_{uv}(t)$ tells the number of the ubiquitous foremost journeys begining at the time t, starting at u, and arriving at v at the time $f_{uv}(t)$.

Once we know the matrix $\mathbf{B}$ we compute

$$n_{uw(v)}(t) = n_{uv}(t)\cdot n_{vw}(f_{uv}(t))$$

if $f_{uw}(t) = f_{vw}(f_{uv}(t))$. Otherwise $n_{uw(v)}(t)$ is equal to $(\infty, 0)$.

6. Conclusion and future work

In the article, we described a new algebraic approach to the analysis of temporal networks that is based on temporal quantities over the selected semiring. We defined a new semiring for computing foremost journeys (first arrival semiring) and traveling semirings in which we can use additional data on the links, besides the latency.

Our description of a temporal network avoids an explicit record of vertex and link presence as it is done in most of the literature. We describe the presence implicitly using the zero in the semiring. Our approach allows more temporal data to be added to the vertices and to the links of the network. In addition to the latency it is possible to add lengths, number of ways, and other temporal information. With the definition of the traveling semiring we can mathematically describe journeys in temporal networks and allow more data in their analysis.

We defined the betweenness centrality with respect to the ubiquitous foremost journeys in temporal networks, and showed how to use the semiring operations to compute it.

For future research other methods from static networks could be generalized and special methods that are adapted to the time dimension should be developed. Also, the definition of betweenness could be generalized or adapted in a way that all the foremost journeys will be taken into account. It seems that for this case, a different semiring should be constructed.

There are still questions about the journeys with zero or fixed waiting times. Both cases raise some interesting questions. Our current solutions fail at the right distributivity. It seems that the fixed time is a very strong assumption and by our intuition it will be difficult to solve.

The procedures for the analysis of temporal networks with zero latency and zero waiting times from our articles Batagelj & Praprotnik (2014); Praprotnik & Batagelj (2015) are available as a Python library TQ (Temporal Quantities) at `http://pajek.imfm.si/doku.php?id=tq`. In the future, we intend to extend the library for the case of temporal networks with latency and arbitrary waiting times.

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References

Baras, J. S. & Theodorakopoulos, G. (2010) Path Problems in Networks. *Synthesis Lectures on Communication Networks*, **3**(1), 1–77.

Batagelj, V. (1994) Semirings for social network analysis. *Journal of Mathematical Sociology*, **19(1)**, 53–68.

Batagelj, V. & Praprotnik, S. (2014) An algebraic approach to temporal network analysis.. *Submitted to Social Networks*.

Bell, M. G. H. & Iida, Y. (1997) *Transportation network analysis*. Chichester: Wiley.

Bhadra, S. & Ferreira, A. (2003) Complexity of Connected Components in Evolving Graphs and the Computation of Multicast Trees in Dynamic Networks.. In Pierre, S., Barbeau, M. & Kranakis, E., editors, *ADHOC-NOW*, volume 2865 of *Lecture Notes in Computer Science*, pages 259–270. Springer.

Carre, B. (1979) *Graphs and networks*. Clarendon Press; Oxford University Press Oxford; New York.

Casteigts, A., Flocchini, P., Quattrociocchi, W. & Santoro, N. (2012) Time-varying graphs and dynamic networks. *International Journal of Parallel, Emergent and Distributed Systems*, **27**(5), 387–408.

Correa, J. R. & Stier-Moses, N. E. (2011) Wardrop equilibria. *Wiley Encyclopedia of Operations Research and Management Science*.

Dolan, S. (2013) Fun with Semirings: A Functional Pearl on the Abuse of Linear Algebra. *SIGPLAN Not.*, **48**(9), 101–110.

Fletcher, J. G. (1980) A More General Algorithm for Computing Closed Semiring Costs Between Vertices of a Directed Graph. *Commun. ACM*, **23**(6), 350–351.

Freeman, L. C. (1977) A Set of Measures of Centrality Based on Betweenness. *Sociometry*, **40**(1), 35–41.

Freeman, L. C. (1978) Centrality in social networks; Conceptual clarification. *Social Networks*, **1**(3), 215–239.

George, B. & Kim, S. (2013) *Spatio-temporal Networks; Modeling and Algorithms*. Springer Briefs in Computer Science. Springer.

Gondran, M. & Minoux, M. (2008) *Graphs, Dioids and Semirings: New Models and Algorithms (Operations Research/Computer Science Interfaces Series)*. Springer Publishing Company, Incorporated, 1 edition.

Holme, P. (2005) Network reachability of real-world contact sequences. *Physical Review E (Statistical, Nonlinear, and Soft Matter Physics)*, **71**(4), 46119.

Holme, P. & Saramäki, J. (2012) Temporal networks. *Physics Reports*, **519**(3), 97–125.

Holme, P. & Saramäki, J. (2013) *Temporal networks. Understanding Complex Systems*. Springer.

Moder, J. J. & Phillips, C. R. (1970) *Project management with CPM and PERT*. Reinhold industrial engineering and management sciences textbook series. Reinhold Pub. Corp., 2 edition.

Mohri, M. (2002) Semiring Frameworks and Algorithms for Shortest-distance Problems. *J. Autom. Lang. Comb.*, **7**(3), 321–350.

Nicosia, V., Tang, J., Musolesi, M., Russo, G., Mascolo, C. & Latora, V. (2011) Components in time-varying graphs. *CoRR*, **abs/1106.2134**.

Praprotnik, S. & Batagelj, V. (2015) Spectral centrality measures in temporal networks. *Submitted to Ars Mathematica Contemporanea*.

Riordan, J. (1958) *Introduction to Combinatorial Analysis*. Dover Books on Mathematics. Wiley New York.

Santoro, N., Quattrociocchi, W., Flocchini, P. & Casteigts, A. (2011) Time-varying graphs and social network analysis: Temporal indicators and metrics. *3rd AISB Social Networks and Multiagent Systems Symposium (SNAMAS*, pages 32–38.

Wasserman, S. & Faust, K. (1994) *Social network analysis: Methods and applications*. Cambridge University Press.

Xuan, B. B., Ferreira, A. & Jarry, A. (2003) Computing shortest, fastest, and foremost journeys in dynamic networks. *International Journal of Foundations of Computer Science*, **14**(2), 267–285.

Zimmerman, U. (1981) *Annals of Discrete Mathematics: Linear and Combinatorial Optimization in Ordered Algebraic Structures*. North-Holland.