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A FORMULA INVOLVING THE NUMBER OF 1-FACTORS IN A GRAPH

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ABSTRACT

A formula connecting the number of 1-factors in some subgraphs of a graph is proved.

A set of independent edges that cover all vertices of a graph is called a 1-factor of that graph. The number of 1-factors of a graph G is denoted by $K(G)$, with $K(G) = 1$ if G has no vertices.

Proposition 1. If the edge u of a graph G with an even number of vertices joins the vertices x and y, then

$$K(G-x-y)\cdot K(G-u) = \sum_{Z} (K(G-Z))^2 \quad , \tag{1}$$

where Z is a circuit of G and the summation on the r.h.s. of (1) runs over all even circuits of G containing the edge u.

Proof. Let A and B be the sets of 1-factors of G which contain and which do not contain u. Then

$$|A| = K(G-x-y) \quad , \quad |B| = K(G-u) \quad .$$

If $a_i \in A$ and $b_j \in B$, then $c_{ij} = a_i \cup b_j$ contains an even circuit Z passing through u. Both $a_i' = a_i \setminus Z$ and $b_j' = b_j \setminus Z$ are 1-factors of $G-Z$. The number of c_{ij}'s containing Z is equal to the number of ordered pairs (a_i', b_j'), i.e. $(K(G-Z))^2$. Since the total number of c_{ij}'s is equal to $|A||B|$, we get (1).

This completes the proof.

Formula (1) has been proved in [1] for hexagonal animals and its validity is now extended to all graphs with an even number of vertices.

Let now G be a graph with an odd number of vertices. Subdividing the edge u with the new vertex z we obtain from G the graph $G(u/z)$. Let us look for a number of 1-factors in $G(u/z)$. There are two possibilities represented in the Fig.1.

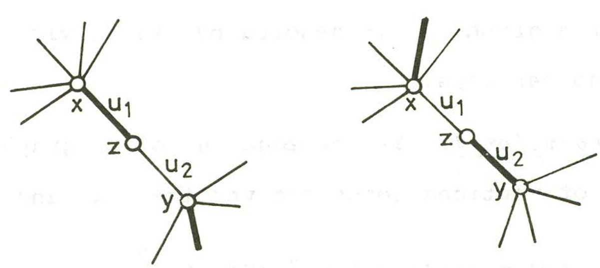

Fig.1.

It is evident that

$$(2)\qquad \begin{aligned} K(G(u/z)-x-z) &= K(G-x) = K(G(u/z)-u_2) \;, \\ K(G(u/z)-y-z) &= K(G-y) = K(G(u/z)-u_1) \;. \end{aligned}$$

By (1) it follows also

$$(3)\qquad K(G(u/z)-x-z)\cdot K(G(u/z)-u_1) = \sum_Z (K(G(u/z)-Z))^2 \;,$$

where Z runs over all even circuits of $G(u/z)$ containing the edge u_1; and from (3) considering (2) we finally obtain the following proposition.

Proposition 2. If the edge u of a graph G with an odd number of vertices joins the vertices x and y, then

$$(4)\qquad K(G-x)\cdot K(G-y) = \sum_Z (K(G-Z))^2 \;,$$

where Z runs over all odd circuits of G containing the edge u.

Because graphs with odd number of vertices have no 1-factor we can combine (1) and (4) in the following theorem.

Theorem. If the edge u *of a graph* G *joins the vertices* x, y, *then*

$$K(G-x)\cdot K(G-y) + K(G-x-y)\cdot K(G-u) = \sum_Z (K(G-Z))^2 \;,$$

where Z *runs over all circuits of* G *containing the edge* u.

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[1] I. Gutman, *Topological properties of benzenoid systems. VI. On Kekulé structure count*, *Bull. Soc. Chim. Beograd* 46 (1981), 411-415.